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MATHEMATICS MAGAZINE

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AUTHORS

Steven J. Brams is professor of politics at New York University. He is a fellow of the AAAS and was the recipient of a Guggenheim fellowship. He is the author or co-author of twelve books involving applications of game theory and social choice theory, to voting and elections, international relations, and the Bible and theology. His most recent books are *Theory of Moves* (Cambridge University Press, 1994) and, with Alan D. Taylor, *Fair Division: From Cake-Cutting to Dispute Resolution* (Cambridge University Press, 1996).

David Jeffrey received his B.Sc. from Monash University in Australia. His Ph.D. was obtained at Cambridge University in 1974. His research has led him, somewhat unintentionally, into the community of computer algebra system developers. This paper is a digression from an algorithm (sadly not a formula) for integrating rational trigonometric functions; do not ask him how long he spent trying to get a cute formula for the sixth-degree case.

D. Marc Kilgour is professor of mathematics at Wilfrid Laurier University in Canada, and also has appointments in strategic studies and systems engineering, the latter at the University of Waterloo. He is coauthor, with Brams, of *Game Theory and National Security* (Basil Blackwell, 1988) and, with L. Fang and K. W. Hipel, of *Interactive Decision Making: The Graph Model of Conflict Resolution* (Wiley, 1993).

James Propp completed his A.B. at Harvard in 1982 and his Ph.D. at Berkeley in 1987. He did three years of postdoctoral work before coming to the Massachusetts Institute of Technology. His research interests are in combinatorics, probability, dynamical systems, and games. For several years he has run an undergraduate research group on the study of tilings of finite regions; more information is available from `http://web.mit.edu/tiling/www/`. James is the author of the forthcoming book *Who Proved Fermat's Theorem?: The Curious Incident of the Boasting Frenchman.*

Bariş Tan received a B.S. degree in 1990 from Bǒgaziçi University, in Istanbul, Turkey. While continuing his graduate work at the University of Florida and playing Monopoly in his spare time, he decided to work on stochastic modeling after reading the article "Monopoly as a Markov Process" in *Mathematics Magazine*. After receiving his Ph.D. in operations research from the University of Florida in 1994, he joined Koç University Business School in Istanbul. Currently he teaches operations research and plays Monopoly, RISK, and other board games. While his primary research area is stochastic modeling of production systems, he finds stochastic modeling of board games more fun and also more useful for himself.

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ARTICLES

The Truel

D. MARC KILGOUR
Wilfrid Laurier University
Waterloo, Ontario N2L 3C5, Canada

STEVEN J. BRAMS
Department of Politics
New York University
New York, NY 10003

1. Introduction

A truel is a three-person expansion of a duel, in which each of the players can fire bullets at the others to try to eliminate them and survive itself. (We will use the impersonal "it" to refer to a player, both to avoid gender distinctions and to admit the possibility of non-human players.) In duels, the interest of game theorists lies in the nature and timing of each player's actions against its opponent. In truels, another strategic consideration comes into play—namely, that a player must also decide which, if either, opponent to fire at, bringing in the target aspect of a confrontation.

Typically in game theory, the rules of a game are specified, and then their consequences for optimal play are derived. We shall discuss such consequences for truels with different sets of rules and, in addition, analyze truels in which some of the rules, such as the length of play, are not specified in advance, endowing players with greater freedom of choice. Sometimes, as we shall see, this enhanced freedom helps the players achieve their goals, which in a truel may be accomplished by abstaining from firing.

The first mention of truels we know of is in Kinnaird's **[13]** compendium of mathematical puzzles, though the word "truel," coined by Yale University economist Martin Shubik [**16**, pp. 22–25], does not appear until the 1960s. The truel has also had other incarnations, including players throwing darts at balloons representing other players; when a balloon is burst, the corresponding player is eliminated from the competition [**14**].

In the animal world, the renowned ethologist Konrad Lorenz [**15**] has noted that it is not uncommon for three fierce rivals to live in proximity and yet eschew attacking each other. Recent truels in films give a more mixed picture: the climactic scenes in Quentin Tarantino's two films, *Reservoir Dogs* (1992) and *Pulp Fiction* (1995), are truels, but the outcomes are very different in each. In a real-life case of competition among the three major television networks in 1992, ABC effectively fired in the air by sticking with its popular late-night news show, *Nightline*, which forced CBS and NBC into a duel over which comedian, David Letterman or Jay Leno, to try to hire in order to capture the late-night TV entertainment audience [**7**].

To facilitate the study of target selection in truels, think of three players, A, B, and C, as standing at the vertices of an equilateral triangle. Each player, while hoping to survive itself, would like to shoot an opponent at one of the other two vertices.

Consider three possible *firing rules*:

1. *Sequential (fixed order)*: The players fire one at a time in a fixed, repeating sequence, such as A, B, C, A, B, C, A... .
2. *Sequential (random order)*: The first player to fire, and each subsequent player, is chosen at random among the survivors.
3. *Simultaneous*: All surviving players fire simultaneously in every round.

In some formulations of the truel, it is possible to fire in the air and thereby not try to eliminate an opponent. Indeed, this strategy may be optimal under firing rule 1: if the firing order is A, B, C, and each player has only one bullet and is a perfect shot, A should fire in the air, eliminating itself as a threat to the other two players; then B will shoot C, leaving A and B as the two survivors. If, on the other hand, A had shot B at the start, C would then shoot A, leaving C as the only survivor. Likewise, if A begins by eliminating C, B would then shoot A, so A does better to miss intentionally (for example, by firing in the air) to ensure its survival as well as B's survival.

This scenario illustrates that there may be more than one survivor in a truel, which occurs in this case because the survivors run out of bullets. But even if the players have an unlimited supply of bullets, the truel may still terminate with more than one survivor because no player wants to be the first to fire. Under firing rule 1, for example, A will have no incentive to try to eliminate an opponent, because whether A eliminates B or C, the other survivor will subsequently shoot A. Consequently, no player will shoot at another, so all will survive despite the fact that the ammunition supply is unlimited.

Note that there must be at least one survivor under firing rules 1 and 2, because only one player fires at a time. By contrast, under firing rule 3, it may be that nobody survives if, say, A fires at B, B fires at C, and C fires at A.

So far we have implicitly assumed that all players in a truel desire to live. We will make this goal explicit:

• *each player prefers an outcome in which it survives to one in which it does not survive.*

An *outcome* of a truel is a subset of {A, B, C}. Under firing rule 3, any subset, including the empty subset, is possible. Under firing rules 1 and 2, the subset must be nonempty unless a player shoots itself, but suicide is contrary to our postulated goal of survival.

2. Better Marksmanship Can Hurt

Almost all mathematical research on truels concerns the relationship between a player's marksmanship (the probability that it hits its target) and its survival probability. Marksmanship, we suppose, does not depend on the particular opponent at which one fires; one is an equally good shot against either opponent. Let the marksmanships of A, B, and C be a, b, and c, respectively, and consider a sequential truel in which the players are not allowed to fire in the air.

Now a player who is about to fire, whether its turn comes up in a fixed order or is chosen at random, has one strategic choice—at whom to fire. To make this choice optimally, we assume that a player considers only its survival probability.

Regardless of what the other players do, a player in a sequential truel maximizes its survival probability by firing at the opponent against whom it would *less* prefer to fight a duel. The reason is simple: if a player's shot misses, then it makes no difference who was the target; but if a shot hits the target, the shooter is better off, because its opponent in the next duel is weaker. Thus, a player who follows this *stronger-opponent strategy* fires at the opponent whose marksmanship is highest, trying to eliminate this player so as to set up a duel with the lower-marksmanship player [**9**, **10**, **11**].

Assume that the marksmanships satisfy $a > b > c$. Then if all three players are alive, A will optimally fire at B, and both B and C will fire at A. Depending on the actual values of a, b, and c, all orderings of the resulting survival probabilities, P_A, P_B, and P_C, are possible. For instance, in an unlimited-supply sequential random-order truel with $a = 0.8$, $b = 0.6$, and $c = 0.4$, the players' survival probabilities are $P_A = 0.296$, $P_B = 0.333$, and $P_C = 0.370$. (These figures are based on the assumption that players continue to fire until only one survives, which will occur eventually as the number of shots fired approaches infinity.)

What is surprising here is that the survival probabilities are in the reverse order of the marksmanships: the worst shot (C) is mostly likely to survive, and the best shot (A) least likely.[1] However, for other marksmanship values, the survival probabilities of A, B, and C might end up in any order, as shown in Table 1, illustrating the sensitivity of the outcome to the quantitative values.

TABLE 1 Possible orderings of survival probabilities in sequential (random-order) truel

Marksmanship			Survival Probabilities			Ordering of Survival Probabilities
a	b	c	P_A	P_B	P_C	
0.9	0.5	0.1	0.540	0.333	0.127	$P_A > P_B > P_C$
0.9	0.5	0.3	0.397	0.294	0.309	$P_A > P_C > P_B$
0.8	0.7	0.2	0.376	0.412	0.212	$P_B > P_A > P_C$
0.6	0.5	0.25	0.314	0.370	0.316	$P_B > P_C > P_A$
0.8	0.5	0.4	0.314	0.294	0.392	$P_C > P_A > P_B$
0.8	0.6	0.4	0.296	0.333	0.370	$P_C > P_B > P_A$

For example, if $a = 0.9$, $b = 0.5$, and $c = 0.1$, even though both B and C fire at A as long as they and A survive, the marksmanships of B and C are too low to boost their survival probabilities above A's. Similarly, C's marksmanship, compared with B's, is also too low to put C's survival probability above B's—even though A will begin by shooting at B—so the order of survival probabilities in this case duplicates the order of the marksmanship of the three players.

3. The Instability of Pacts

Consider a sequential (fixed-order) truel in which the players have unlimited ammunition. The firing order is A, B, C, and the players repeat in this order until only one player survives.

[1] The odd notion that poor marksmanship can contribute to one's chances of survival was recognized in the early literature on truels [**14**].

If $a = 0.8$, $b = 0.7$, and $c = 0.6$, this is a highly competitive truel. Yet when the players choose their (optimal) stronger-opponent strategies, their survival probabilities neither mirror their marksmanships nor are they close in value: $P_A = 0.285$, $P_B = 0.075$, and $P_C = 0.640$.

The biggest boost to C's probability comes when A eliminates B on the first shot with probability 0.8, and then—with B out of the picture—C eliminates A with probability 0.6, giving a joint probability of 0.48 that C alone survives after two shots. B's biggest boost comes when A misses B (0.2), B hits A (0.7), C misses B (0.4), and B hits C (0.7). However, this chain of events—the shortest possible in which B alone survives—has a joint probability of less than 0.04. In summary, C benefits enormously from not being a target until either A or B is killed.

Now suppose that A and B make a pact both to fire at C, rather than at each other, until C is eliminated. Then $P_A = 0.331$, $P_B = 0.661$, and $P_C = 0.008$, which lowers P_C drastically (hardly any surprise) while giving B a substantial edge over A (because C will also fire at A as long as both survive). It would be in the interest of not only B, but also A, to make such a pact, because both do better under it ($P_A = 0.331$ and $P_B = 0.661$) than following the stronger-opponent strategy of firing at each other until one is killed ($P_A = 0.285$ and $P_B = 0.075$).

But who is to say that, after concluding such a pact, the players wiill abide by it? (The rules, as we have formulated them, do not specify a way of enforcing the pact.) In fact, it is not difficult to show that it would be in the individual interest of both A and B to agree to make C the common target—and then to renege on the agreement.

We have carried out the calculations for two cases. First, suppose that it is impossible for a player to detect the target of a shot unless the shot actually hits its target. This means, for instance, that if A agrees to fire at C, but instead fires at B, then B will not know that A has broken the agreement unless B is actually eliminated by A's shot, which is too late, of course, for B to retaliate. We call this case the *target-undetectable truel*, which is analogous to a "silent duel," except that the silence in a duel—caused by silencers on the guns—is over whether a shot has been fired, not who, as in the truel, is the target.

The upper half of FIGURE 1 gives a 2 x 2 payoff matrix for A and B in the case of the target-undetectable truel. In this figure, A and B each have two strategies: Abide by Pact (as long as all three players are alive), or Stronger-Opponent Strategy. The lower-left entries in each cell are the survival probabilities for the row player (A), the upper-right entries the survival probabilities for the column player (B), and the circled entries the survival probabilities of the third player, C.

Notice that Stronger-Opponent is *strictly dominant*, or unconditionally better, for both players: A always does better choosing it, regardless of what B does; similarly, B always does better choosing it, too. In addition, C does best when both A and B choose Stronger-Opponent ($P_C = 0.640$), and worst when they both choose Abide ($P_C = 0.008$). (In this calculation, we assume that C always chooses its strictly dominant strategy, Stronger-Opponent, which is to fire at A as long as A is alive.)

The 2 x 2 game between A and B is actually the infamous Prisoners' Dilemma game in a new guise. In the original game, two prisoners have an incentive to squeal on each other after agreeing not to confess; in the truel, A and B have an incentive to double-cross each other after agreeing to make C their common target.

Our second case, that of the *target-detectable truel*, occurs when the players can identify the target of a shot that missed, which is analogous to the "noisy duel" in which who fired a shot and when is known immediately, even if the shot misses. For instance, if A adheres to the pact but B does not, then if A misses, B will fire at A rather than C, and A will know this even if B also misses. When this happens, C will

1. *Target Undetectable*

A \ B	Abide by Pact	Stronger-Opponent Strategy
Abide by Pact	.6611 .3314 .0075	.6854 .2093 .1053
Stronger-Opponent Strategy	.0507 .4072 .5421	.0750 .2851 .6399

2. *Target Detectable*

A \ B	Abide by Pact	Stronger-Opponent Strategy
Abide by Pact	.6611 .3314 .0075	.6707 .2110 .1183
Stronger-Opponent Strategy	.0750 .2851 .6399	.0750 .2851 .6399

Key: Survival Probability of A at lower left, of B at upper right, of C circled. C assumed to follow Stronger-Opponent Strategy.

FIGURE 1

shoot next, firing at A. If C also misses, then A can fire at B in the second round. This helps A only slightly, however, because the probability that all three players will miss in the first round is only 0.024.

Because A fires first, the detectability of targets implies that B will find out immediately if A reneges on the pact. If this happens, B's best choice is to fire at A, which it can do in the first round.

The lower half of FIGURE 1 shows that the strategic situation changes when the players have target-detection capability. Notice that B's Stronger-Opponent Strategy is *weakly dominant*: when A chooses Abide, Stronger is better for B; but when A chooses Stronger, B's survival probabilities are the same, 0.075 (this tie is what prevents Abide from being strictly dominant).

As for A, Abide is better when B chooses its Abide Strategy, whereas A's Stronger-Opponent Strategy is better when B chooses its Stronger-Opponent Strategy. Anticipating that B will choose its weakly dominant Stronger-Opponent Strategy, however, A will choose Stronger, too, leading to worse payoffs for both players ($P_A = 0.285$ and $P_B = 0.075$) than had they both chosen Abide ($P_A = 0.331$ and $P_B = 0.661$).

Thus, the strategic situation in this case results in an outcome as grim as in the target-detectable case, even though the game is not Prisoners' Dilemma. Unless the players can trust each other in the absence of an enforceable agreement, both suffer when they (rationally) renege on their pact.

4. Perfect Marksmanship and Additional Goals

We next shift the focus from marksmanship, and its specific numerical consequences, to the effects of rules, including those that impute goals to players as well as distinguish among them. Henceforth we assume that all players have marksmanship 1—that is, each player is a perfect shot and, therefore, eliminates with certainty whichever opponent it chooses as a target. We also assume that, in addition to its primary goal of survival,

• *each player prefers an outcome at which fewer of its opponents survive*

as a secondary goal.

To define a tertiary goal, we suppose that each player has one opponent who is its *antagonist*—the opponent it dislikes more—and one who is not. If the antagonist of A is B, we say Ant(A) = B. We assume, as a tertiary goal, that

• *each player, when exactly one of its opponents survives, prefers an outcome at which that surviving opponent is not its antagonist (whether the player survives or not).*

As soon as Ant(A), Ant(B), and Ant(C) have been specified, the preference rankings of all three players are strictly determined. For example, if Ant(A) = B, then A's preference ranking in descending order is

$$\{A\}, \{A, C\}, \{A, B\}, \{A, B, C\}, \varnothing, \{C\}, \{B\}, \{B, C\},$$

where $\varnothing$ is the empty set (no survivors).

5. Simultaneous Truels: More Ammunition Can Help

So far we have assumed sequential rules, except to note that when firing is simultaneous, any number, including none, of the players may survive. However, given the assumption that all the players are perfect shots, then a simultaneous truel, without the option of firing in the air, cannot produce two or three survivors: either one player survives, or none does, because at least two players are always eliminated when the players' shots are directed at each other.

Now consider the effects that antagonists can have on outcomes. If firing in the air is possible, and there is one player who is not the antagonist of any other player, then that player, knowing that its two opponents will eliminate each other (since they are mutual antagonists), can either fire at its antagonist or into the air (if this is permitted). In either case, the nonantagonist is rewarded by being the only survivor.

The preceding analysis applies to the one-bullet simultaneous truel. If there is more than one bullet, and firing in the air is possible, then it is possible that a duel, which we assume to have the same simultaneous firing rule, results after the first round.

Suppose that the surviving players are A and B on the second round. Whoever their antagonists are, they would prefer the outcome {A, B} to $\varnothing$. But $\varnothing$ is what they will get, because each player does better firing at its remaining opponent, whether that opponent returns fire or not, which is another example of a Prisoners' Dilemma: the mutually preferred outcome, {A, B}, falls prey to the mutually less preferred outcome, $\varnothing$, because the noncooperative strategy of each player in the duel strictly dominates its cooperative strategy, producing a better outcome whatever its opponent does.

Nevertheless, there is hope in a situation like this, but it requires foresight on the part of all the players. Suppose that, on the first round, the mutual antagonist of A and B, C, was shot by both, while C fired in the air. If each player had only one bullet, then {A, B} would be the outcome.

But now suppose that the players have more than one bullet each. Then, as we just argued in the previous paragraph, A and B will fight a duel, and the outcome will be ∅. Rationally anticipating this situation, however, A and B would have good reason to fire in the air on the first round, because they prefer the outcome {A, B, C} to ∅. In fact, as long as the players have an unlimited supply of bullets, all three should fire one in the air on each round, thereby reducing their arsenals.

Alternatively, if they are allowed to "pass" a round without firing a bullet, then a supply of at least two bullets to each player will suffice to keep the players from ever firing at each other if they are farsighted. For as soon as one player shoots another, it and the surviving player can anticipate a duel, which will be fatal to them. If they want to avoid ∅, therefore, all players will pass on every round. Consequently, *more* ammunition can foster mutual deterrence in a simultaneous truel.

6. Odd-Even Effects in Sequential Truels

In a sequential truel with firing order A, B, C, A, B, C, A..., assume each player is a perfect shot and has one bullet. Each player can either fire at another player or not fire. Suppose that the antagonists of the three players are the following: Ant(A) = B, Ant(B) = C, and Ant(C) = A, or, for short, Ant(A, B, C) = BCA. Then truels of finite length can be analyzed, in sequence, using "game trees," as illustrated in FIGURE 2.

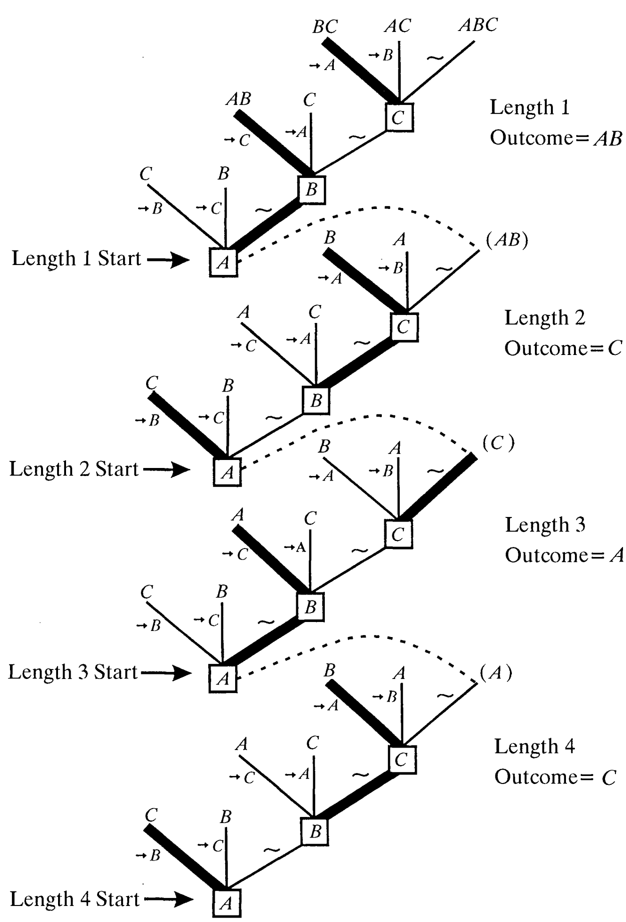

FIGURE 2

A *game tree* describes the set of choices available to each player when its turn comes up in the sequence. Thus, for a game of length 1, A begins by choosing among three options: shoot B ($\rightarrow$ B), shoot C ($\rightarrow$ C), or not fire ($\sim$). If A shoots B, C can then shoot A, making the outcome {C}, or simply C as shown in FIGURE 2. Likewise, if A shoots C, the outcome will be B.

The game goes to a second stage if A does not fire, giving B the three options shown. For example, if B shoots C, then the survivors will be A and B, so the outcome will be {A, B}, or simply AB as shown in FIGURE 2.

Finally, the game goes to a third stage if B does not fire after A also does not, leaving C the final choice. C can either shoot one of its two opponents, leaving two survivors, or not fire, which would enable all three players to survive. All the choices of A, B, and C are shown at the top of FIGURE 2, beginning at "Length 1 Start" and proceeding upward.

One Round. To determine optimal choices in a length 1 game, we work backward from C's final choices in the third stage. If C should survive until this stage, it would prefer to shoot its antagonist A, yielding the outcome BC, rather than choose one of its other two options, so we darken branch $\rightarrow$ A to indicate that this branch would be chosen.

Working backward to the second stage, B can anticipate that its choice of branch $\sim$ would result in BC, which it compares with the outcomes of choosing $\rightarrow$ C (AB) and $\rightarrow$ A (C, because C would then shoot B). Preferring the outcome AB to either BC or C, it would choose $\rightarrow$ C if it should survive the first stage.

Working backward to the first stage, A can anticipate that its choice of branch $\sim$ would result in AB, which it compares with the other two outcomes it can effect, C and B. Preferring AB to these, it would choose branch $\sim$, and B would in turn choose branch $\rightarrow$ C, making the rational outcome of the game AB.

In effect, we worked backward from the top of the game tree (third stage) to the bottom (first stage) to determine the players' rational choices (darkened branches) at each stage. This reasoning process to determine rational choices is called *backward induction*.

Having worked backward from the top of the game tree (third stage) to the bottom (first stage) to determine the players' rational choices (darkened branches) at each stage, we reverse the process to determine what choices the players would actually make. Starting from the bottom of the tree and following the darkened branches upward, we see that the play will never reach the third stage, when it is C's turn to choose, because after A chooses $\sim$, B will choose $\rightarrow$ C, eliminating C from play and giving AB in the one-round game.

We next describe the general solution to the truel for games of any finite length greater than 1. It turns out that there is always only one survivor, not the two (A and B) we just found in a length 1 game.

More Than One Round. What we have just done for a length 1 game we can do for games of length $2, 3, 4, \ldots$, which add successive rounds of play to the game of length 1. (A *round* comprises the three stages of the length 1 game we just analyzed.) Their analysis simply takes the rational outcome of a length k game and substitutes it as the outcome of the first round of a length $k+1$ when all players choose $\sim$ (these substitutions in FIGURE 2 are indicated by the dashed lines).

For example, we know from the foregoing analysis of a game of length 1 that if nobody shoots in the first round of the length 2 game, the rational outcome will be AB in the second round, because this one remaining round is a length 1 game. Consequently, in the first round of the length 2 game, the outcome of branch $\sim$ for C in the third stage is AB (the outcome of the length 1 game) rather than ABC (the outcome, if nobody shoots, of the length 1 game). Backward induction in the length 2 game shows that the rational outcome is C, as shown in FIGURE 2.

Substituting C as the outcome of branch $\sim$ for C in the third stage of the length 3 game yields A as the outcome of this game. Substituting A as the outcome of the branch $\sim$ for C in the length 4 game yields C as the outcome of this game. In summary, games of length 1, 2, 3, and 4 have as outcomes AB, C, A, C, respectively; the $C - A$ alternation continues to repeat for longer-length games, with C as the outcome of all even-length games and A as the outcome of all odd-length games longer than one round.

This is the most extreme sensitivity of game outcomes to the rules that we have discovered. Not only does this truel have three different outcomes (AB, C, and A), depending on how many rounds are played, but it also does not "settle down" as play continues indefinitely because of the even-odd alternation.[2] Technically, this truel has no limiting outcome as the number of rounds approaches infinity.

By contrast, for all other possible antagonisms of the three players (each player has two possible antagonists, so there are $2^3 = 8$ possible antagonisms, as shown in Table 2), the outcome of the truel does not depend on its length, once past the first round. Observe that for the six antagonisms in which there is one player who is nobody's antagonist, that person is invariably the only survivor, underscoring the value of not having enemies.[3]

7. The Problem of Anticipation

The sequential truel for $\text{Ant}(A, B, C) = BCA$ has another curious feature besides its even-odd alternation: whatever its length, all shooting occurs in either the first round, or in the first two rounds. If the truel is of length 1, A does not fire and then B shoots C. If the truel is of length 2 (or any other even length), A shoots B and then C shoots A in the first round. If the truel is of length 3 (or any other odd length except length 1), A does not fire and B shoots C in the first round; then A shoots B in the second round.

While there is never any shooting in rounds $3, 4, \ldots,$ it is the *anticipation* of these subsequent rounds—in particular, whether the total number of rounds to be played is

[2]The sensitivity of outcomes to parity considerations was first noted, as far as we know, by Kilgour [**12**], who showed that two-person games have this sensitivity.

But parity also matters in the following n-person game, which seems to be part of the folklore of game theory. There are n lions in a clearing in the jungle, along with one dead lamb, and the lions are ranked from L1 (highest) to Ln (lowest). The lions move sequentially, in order of rank, and they can choose to eat or not eat. They are *hungry*, and therefore prefer to eat, but they are also *cautious*: they will not eat if eating will lead to their death. The lions have reason to be fearful, because they are *narcoleptic*, *cannibalistic*, and *cowardly*: if they eat, they fall asleep immediately, at which time they will be prey to the next lion in the sequence, who will eat only sleeping lions (or a dead lamb). Finally, the lions are *finicky*, so they will eat only recently dead, or newly asleep, meat—in other words, they will not eat meat that has been passed over by others. It is easy to show that if n is odd, L1 will eat the lamb (safely) and the others will not eat, lest they in turn be eaten. If n is even, L1 will not eat the lamb, and therefore no one else will eat. We owe this formulation of the lion problem to Jeffrey R. Lax.

[3]George Bush and Bill Clinton were each other's antagonists in the 1992 US presidential election, putting Ross Perot in the role of the nonantagonist. This is perhaps a partial explanation—another being Perot's massive campaign spending—of why Perot received a larger percentage of the popular vote (19%) than any third-party candidate since Theodore Roosevelt, who received 27% in 1912 (Roosevelt had previously been president).

TABLE 2 Dependence of outcome on length and antagonisms in the sequential (fixed-order) truel

Ant(A,B,C) =	BCA	CAB	BAA	CAA	BCB	BAB	CCA	CCB
Length = 1	AB	AB	C	B	AB	AB	AB	AB
Length = 2	C	B	C	B	A	C	B	A
Length = 3	A	B	C	B	A	C	B	A
Length = 4	C	B	C	B	A	C	B	A
Length = 5	A	B	C	B	A	C	B	A

even or odd—that completely determines whether the rational outcome is C (even) or A (odd). However, if the game can continue indefinitely, with nobody able to predict exactly when it will end, then outcome ABC is a rational possibility. Because the players will never know for sure if they are in a last-round situation, no player will fire at any other, so all will survive.[4]

This is the result we discussed at the beginning of this article for a sequential (fixed-order) truel without antagonisms, in which each player had one bullet and was a perfect shot. But it applies equally well if there is no fixed order, and the players have more than one bullet. The reason is that if any player shoots an opponent, the third player will then shoot the shooter, so nobody wants to get in the first shot lest it become the next victim.

To return to the alternation result, it is incredible to us that a truel that goes, say, 64 rounds will have a completely different outcome from one that goes 65 rounds. While the logic of backward induction is impeccable, it seems highly questionable that real-life players—chess grand masters notwithstanding—calculate and act upon it.[5]

At the same time, we would not dismiss this logic out of hand, because players in real-life games appear to use it. But they use it in simplified form, sometimes attempting to look only three or four steps ahead [**4**], sometimes an indefinite number [**8**], or sometimes looking for a generalization that seems applicable ("never be the first to shoot"). Indeed, part of the "game" may be to define its rules, such as the sequencing of moves, making real-life truels usually less structured and more free-flowing than the abstract and somewhat ethereal ones we have analyzed.

[4]A more precise (probabilistic) formulation of "indefinite continuation" is given in Brams and Kilgour [**6**], wherein we show that the even-odd result and the all-survive result can coexist as equilibria, provided that the players view the truel as "likely enough" to continue at least one more round. (Although there can be no coexistence if the truel is bounded—that is, it is certain to end after a fixed number of rounds—coexistence can occur when the truel, while finite, is unbounded, so an upper bound on when it will end cannot be specified.) This coexistence of equilibria, we argue, puts the players in a quandary, whereby their expectations about which equilibrium will occur determine whether they shoot from the start, leaving only one survivor (or two, if the truel ends after one round), or withhold their fire, ensuring that all survive.

[5]The role that backward induction, and common knowledge of it, plays in the solution of perfect-information games remains controversial, as a recent exchange shows [**1**, **2**].

8. The Reality of Truels

We believe that what was called "extended" nuclear deterrence during the Cold War —the US threat to retaliate against the Soviet Union if it, say, invaded West Germany [**3**, **5**]—worked to protect Western Europe against a Soviet attack because the Soviet Union was the common antagonist of the other two players. In effect, US participation protected Western Europe; in the absence of the US, the European confrontation could have become a duel in which the duelists, Western Europe and the Soviet Union, might have found themselves in a shoot-out.

In the Yugoslav conflict in the early 1990s, the Bosnian Serbs, by seizing territory in the beginning, made themselves the common antagonist of the Bosnian Muslims and Croats. This unlikely coalition struck back in later rounds, though we are hasten to add that this conflict is far too complex to analyze as a simple truel (for one thing, the UN and NATO constitute significant other players).

Nevertheless, some of the features we have explored in abstract truels, under different rules, provide rough analogies to three-person conflicts—in which there may be an attrition of players—in the world today. The hard part is to identify the rules under which these conflicts are played out, which may be inchoate or which the players themselves may have an opportunity to manipulate.

As we have seen, optimal play can be very sensitive to slight changes in the rules, such as the number of rounds of play allowed. At the same time, some findings for truels are quite robust: the weakness of being the best marksman, the fragility of pacts, the possibility that unlimited supplies of ammunition may stabilize rather than undermine cooperation, and the deterrent effect of an indefinite number of rounds of play (which can prevent players from trying to get in the last shot).

Some of these findings are counterintuitive, even paradoxical [**6**]. An understanding of them, we believe, might well dampen the desire of aggressive players to score quick but temporary wins, rendering them more cautious. In particular, contemplating the consequences of a long and drawn-out conflict, truelists may come to realize that their own actions, while immediately beneficial, may trigger forces that ultimately lead to their own destruction.

Acknowledgments. We thank two anonymous referees for their valuable comments on an earlier version of this paper. D. Marc Kilgour gratefully acknowledges the support of the Laurier Centre for Military Strategic and Disarmament Studies and the Social Sciences and Humanities Research Council of Canada, and Steven J. Brams the support of the C. V. Starr Center for Applied Economics at New York University.

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A Response to "Bell's Conjecture"

Dear Editor:

The poem "Bell's Conjecture" (this MAGAZINE, June 1997, p. 203) adopted E. T. Bell's ranking of Archimedes, Newton, and Gauss as the greatest mathematicians of all time. We felt compelled to respond to the omission of Leonhard Euler from such glorious company. Thus...

Before we let you get away,
Your choices set in stone,
Consider what we have to say:
E.T.! 0, please! Call home!

Stop the presses! Hold that thought!
And listen to our voices.
Ruffled, even overwrought,
We'll supplement your choices.

Old Archie, Isaac, C. F. Gauss—
Though each deserves a floor
In mathematics' honored house,
Make room for just one more.

Without the Bard of Basel, Bell,
You've clearly dropped the ball.
Our votes are cast for Euler, L.
Whose *Opera* says it all:

Six dozen volumes—what a feat!
Profound and deep throughout
Does Leonhard rank with the elite?
Of this there is no doubt.

Consider how he summed, in turn
The quite elusive mix
Of one slash n all squared—you'll learn
He got π^2 slash six.

We're shocked we did not see his name
With those you justly sainted.
No Euler in your Hall of Fame?
Your judgment's surely Taine-ted.

It's time to honor one you missed,
To do your duty well.
Add worthy Euler to your list,
And save him by the Bell.

—CHARLIE MARION
LAKELAND H.S.
SHRUB OAK, NY 10588

—WILLIAM DUNHAM
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Consider how he summed, in turn
The quite elusive mix
Of one slash n all squared—you'll
learn
He got π^2 slash six.

We're shocked we did not see his
name
With those you justly sainted.
No Euler in your Hall of Fame?
Your judgment's surely Taine-ted.

It's time to honor one you missed,
To do your duty well.
Add worthy Euler to your list,
And save him by the Bell.

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A Pedestrian Approach To a Method of Conway, or, A Tale of Two Cities

JAMES PROPP
Massachusetts Institute of Technology
Cambridge, MA 02139-4307

Tilings in two dimensions have long fascinated both professional and recreational mathematicians. The appeal is easy to understand: the objects being studied are concrete, and a person can develop useful intuitions about them through hands-on experimentation that verges on play. One special charm of the subject is that questions about tiling often take the form "Can you tile a ... with ... ?", so that an affirmative answer can be embodied in a single picture. The solver of such a problem may have spent hundreds of hours devising the picture, but a reader can verify its validity in a matter of minutes. (See [2] for an example of this.)

In contrast, a negative answer to a tiling problem—that is, the assertion that a certain kind of tiling does not exist—may not be so simple to verify. If the region to be tiled is finite and there are only finitely many allowed tiles, then it is possible in principle to determine the status of that tiling problem (possible versus impossible) by brute-force, examining all possibilities; if all possibilities can be eliminated then the desired tiling does not exist. But checking the validity of such a proof takes nearly as long as constructing the proof in the first place. What's more, the brute-force approach does not allow one to make the jump from isolated cases to infinite families of tiling problems.

Fortunately, there are techniques for expeditiously proving that some tiling problems cannot be solved. One of the cleverest of these is a method invented by John Conway 35 years ago and developed further by Jeff Lagarias [1]. Although the Conway-Lagarias article is cast in the language of combinatorial group theory, the method often produces simple geometric criteria for non-tileability that can be applied to specific problems without knowledge of the algebraic machinery that gave rise to them.

Conway's theory often permits one to construct simple "certificates of non-tileability" for a region. For instance, we will see that FIGURE 7 is in a certain sense a visual proof of the unsolvability of a particular tiling problem. I like to think of it as a "proof-mandala": if one prepares one's understanding in the proper way (by absorbing some preparatory theorems about the set of allowed tiles) and contemplates the picture in the right frame of mind (by noting certain facts about the untiled portion of the region), then one can achieve enlightenment (perceive the futility of striving to tile the region). Putting it less fancifully, FIGURE 7 is the visual culmination of an argument; it clinches a proof of non-tileability once Conway's theory has been used to formulate a specific concrete criterion.

My purpose here is to give you a glimpse of Conway's ideas in as accessible a way as possible, avoiding combinatorial group theory entirely. The price that you will pay for my pedestrian approach is that some of the ideas in the proof of the main theorem will seem to come out of nowhere. I hope that the novelty and power of the method will intrigue you sufficiently that you feel impelled to read the original articles and learn about the algebraic framework that the proof fits into. In addition to Conway and Lagarias' article [1], there is also a paper of Thurston [8] that gives the method a slightly different slant. I will close the article by briefly discussing the more advanced viewpoints that these two articles take.

Tiles, Tilings and Tileability

The tiles we will consider are of the kind known as *polyominoes*. Every polyomino can be formed by taking a union of cells in a square grid, and the number of cells determines whether the polyomino is classified as a *monomino*, *domino*, *tromino*, *tetromino*, or what have you. (For a precise definition of polyominoes, see **[5]** and **[7]**.) The five tetrominoes are shown in FIGURE 1. These shapes have received recent notoriety through the video game Tetris.

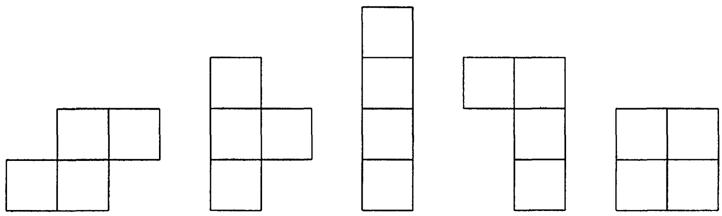

FIGURE 1
The tetrominoes.

When a large polyomino (a union of many cells) can be written as a union of small polyominoes that are disjoint except along their boundaries, we say that the large polyomino has been *tiled* by the small ones. It is easy to show that each of the small polyominoes must be a union of some subset of the cells that jointly constitute the large polyomino.

The book **[7]** discusses many of the tiling properties of tetrominoes, and I will not attempt a general overview here. Instead I am going to focus on one particular sort of tetromino, the so-called *skew tetromino*. This is the tetromino standing at the very left of our family portrait. The skew tetromino cannot be rotated so as to yield its own mirror-image without leaving the plane. Later, one of the skew tetromino's siblings will elbow its way into the action, namely, the *square tetromino*, standing off to the right—but that is getting ahead of our story.

I am going to give criteria for recognizing when a region can be tiled by skew tetrominoes, using all four of the orientations shown in FIGURE 2. These criteria are necessary conditions for tileability, so that when one of them fails, you can conclude that the region in question cannot be tiled. While such criteria can never give you assurance that a tiling problem has an affirmative answer, they can often save you from wasting time looking for a tiling where none exists.

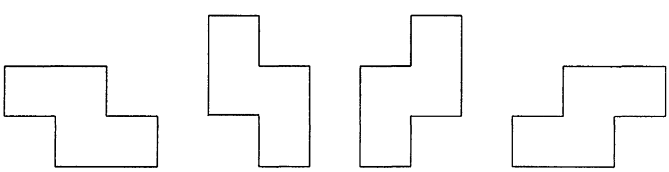

FIGURE 2
The skew tetromino.

One sort of region that will defeat any would-be tiler who has only skew tetrominoes on hand is a rectangle. For consider the skew tetromino that covers the upper-left cell of the rectangle; without loss of generality, we may suppose it is placed as shown in FIGURE 3. The placement of this skew tetromino forces the placement of

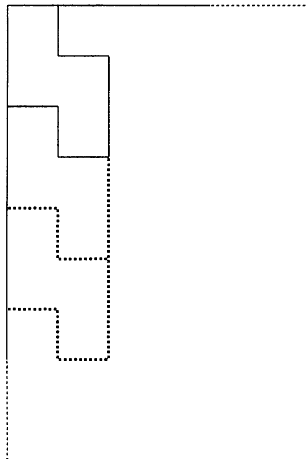

FIGURE 3
Failed tiling of a square.

another skew tetromino below it, and so on, leading to a configuration in the lower-left corner that cannot be extended to a tiling. This is a "quasi-local" obstruction to tiling, in that we only need to look at part of the region to be tiled to deduce untileability.

Such quasi-local arguments for non-tileability fail us when we move on to the tiling problems that are the main subject of this article. Define an *Aztec diamond of order n* as a region consisting of $2n(n+1)$ unit cells, arranged in centered rows of lengths $2, 4, 6, \ldots, 2n-2, 2n, 2n, 2n-2, \ldots, 6, 4, 2$. Aztec diamonds were introduced in **[3]**, where it was shown that the Aztec diamond of order n has exactly $2^{n(n+1)/2}$ tilings by dominoes. FIGURE 4 shows the Aztec diamond of order 5. This region also appears in problem 81 of **[5]**, which asks whether the region can be tiled by the twelve pentominoes in such a way that each pentomino gets used exactly once; this problem

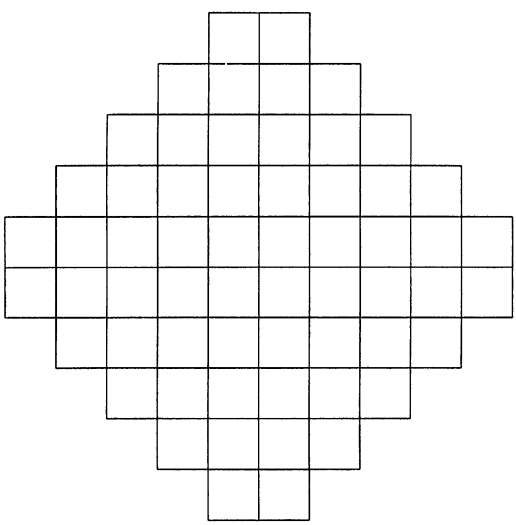

FIGURE 4
Aztec diamond of order 5.

(which also appears as problem 5.22 of [7]) was solved in the negative by Andy Liu [6] ten years ago.

The difficulty with tiling an Aztec diamond is global: FIGURE 5 shows that we can tile a sizable portion of the Aztec diamond of order 17, leaving uncovered only a 6-by-6 block that is far away from the boundary. On the other hand, since (as is easily shown) we can tile the whole plane with skew tetrominoes, we can start tiling our Aztec diamond from the center and work outward until only a narrow fringe along the boundary remains untiled. Thus, it is not difficult to tile any particular portion of the Aztec diamond—what is hard is tiling the whole thing.

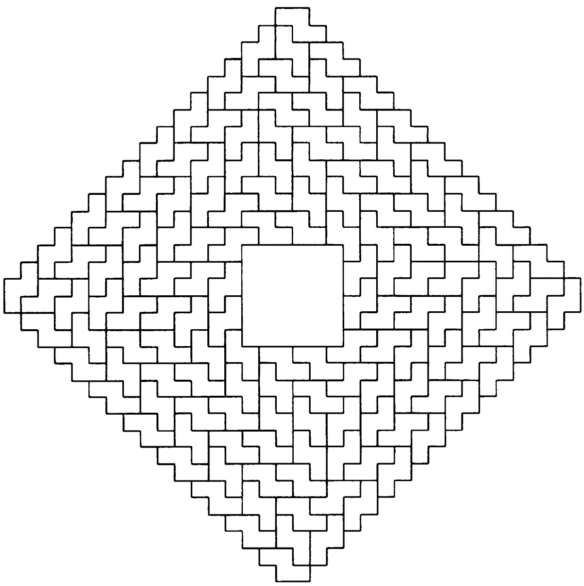

FIGURE 5
Failed tiling of an Aztec diamond.

A well-known technique for proving that particular tiling problems cannot be solved is the use of *coloring arguments*. For instance, to show that the Aztec diamond of order 5 cannot be tiled by skew tetrominoes, impose the coloring shown in FIGURE 6,

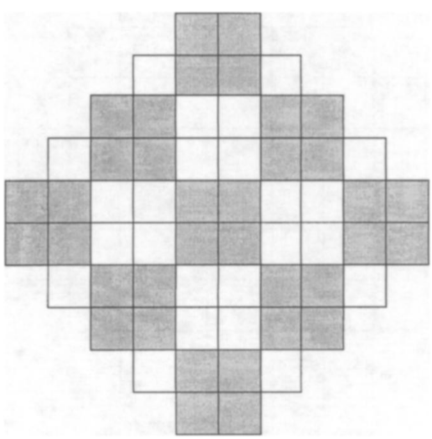

FIGURE 6
Colored Aztec diamond.

with two-by-two blocks of black cells alternating with two-by-two blocks of white cells (along with a few left-over white cells). A skew tetromino of any orientation placed anywhere in the region must cover three black cells and one white cell or vice versa. In particular, it must contain an odd number of black cells. Since any tiling of the entire region by skew tetrominoes must use exactly 15 tiles, and each tile contains an odd number of black cells, the entire region must contain an odd number of black cells. Since the number of black cells in the region is even, we have reached a contradiction, and no such tiling exists.

Suitably generalized, the coloring-scheme of FIGURE 6 tells us that the Aztec diamond of order n cannot be tiled by skew tetrominoes if $n(n+1)/2$ is odd, which

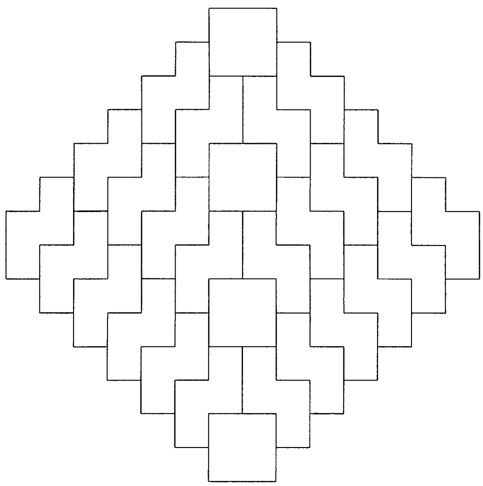

FIGURE 7
The proof-mandala.

happens whenever n is 1 or 2 more than a multiple of 4. But what about values of n for which $n(n+1)/2$ is even? A bit of experimentation should convince you that no tiling exists when $n = 3$ or 4, so one might suppose that a different coloring argument might be devised to handle such cases. However, it can be shown (see **[1]**) that there cannot be a coloring argument (at least in the simplest sense of the term) that proves the untileability of the Aztec diamond of order n for even *one* value of n for which $n(n+1)/2$ is even.

This is where Conway's approach comes to our aid. Following his lead, we will soon see that for *no* positive integer n can the Aztec diamond of order n be tiled by skew tetrominoes. In fact, we will formulate a criterion (the main theorem) that will let us look at FIGURE 7 and, after a moment's inspection, announce that we are satisfied that the Aztec diamond of order 7 cannot be tiled by skew tetrominoes.

Shadowing Paths

To initiate you into the mysteries of such proof-mandalas, I am going to take you on an odd sort of imaginary excursion: you and I will in a sense be walking together, but we will be doing our walking miles apart, in two different cities.

For want of better names, I will call these cities Hoboken and Manhattan, with no slander intended toward either great city. Any resemblance between either of the imaginary places that I will describe and the real places whose names they bear is mostly coincidental.

In Hoboken, all the streets and avenues are one-way, with streets running west to east and avenues running south to north. If you like, you can imagine a two-way ring road around the city that will somewhat alleviate the inconvenience resulting from the city's eccentric traffic system. (See FIGURE 8.)

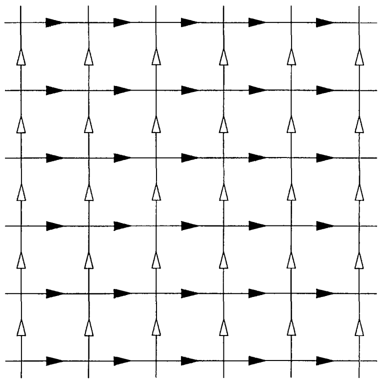

FIGURE 8
Hoboken.

The city is naturally associated with a portion of an infinite square grid; for example, one can interchangeably speak of "vertices" (in the grid) or "intersections" (in the city). To keep track of the traffic pattern, let's mark the edges of the grid with arrows that indicate the direction of traffic (solid arrows for the streets and unfilled arrows for the avenues). Note that each vertex has an ingoing solid arrow, an outgoing solid arrow, an ingoing unfilled arrow, and an outgoing unfilled arrow, and that every intersection looks just like every other.

As a pedestrian in Hoboken, I am free to go with or against vehicular traffic; at each corner I can go in any of the four possible directions. In this way I can trace a path in the grid whose edges connect adjacent vertices. Given my starting point, my journey is uniquely specified by the set of decisions I make as to whether to travel on a street or an avenue and whether to travel with traffic or against it. Conversely, any sequence of such decisions corresponds to an actual path through the city.

Hoboken is the city in which I am going to do my walking; you, however, are going to be walking in Manhattan. This city also has one-way streets and avenues, but the direction of traffic along streets (and along avenues) alternates, instead of being in a single fixed direction throughout the city. (See FIGURE 9.) As before, each vertex has an ingoing solid arrow and an outgoing solid arrow and an ingoing unfilled arrow and an outgoing unfilled arrow (the first two horizontal and the last two vertical). There are four different sorts of intersections in Manhattan, but they are all related to one another by symmetries (reflections and rotations, to be precise).

Say a path in Manhattan *shadows* a path in Hoboken if they are described by the same instructions, in the following sense. As I take my walk in Hoboken, you and I

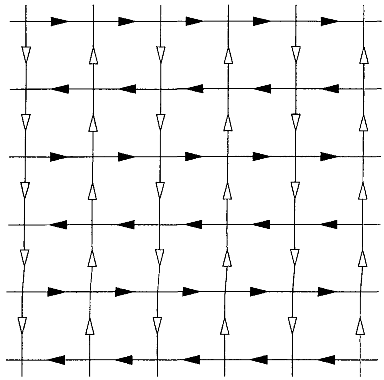

FIGURE 9
Manhattan.

will talk by cell-phone and I will tell you what I am doing: *not* whether I am going north, south, east, or west, and *not* whether I am turning left or right, but merely whether I am going on a street or an avenue, and whether I am going with traffic or against it. If you and I walk at the same pace and you imitate my path as described by me over the cell-phone (which can be done in one and only one way, given your starting point), you are tracing out the *shadow* of my path.

For example, consider the path in the top half of FIGURE 10 that starts at the marked point and traverses the boundary of the skew tetromino in the counterclockwise sense, ending where it started. If I traverse this path in Hoboken, then the description of my path is $S^+S^+A^+S^+A^+S^-S^-A^-S^-A^-$, where S and A denote street and avenue respectively and $+$ and $-$ determine whether I am going with or against traffic. If I describe my path to you in this fashion by cell-phone, and you imitate it in Manhattan,

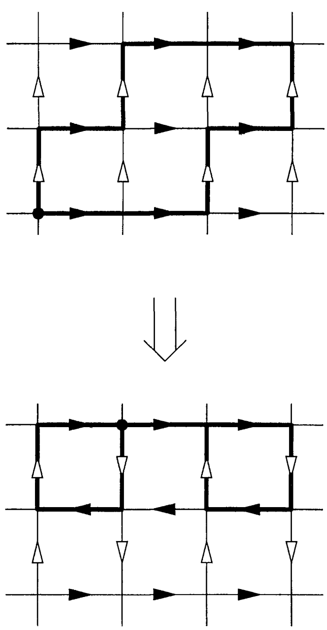

FIGURE 10
Shadowing a path.

you will travel along the path in the bottom half of FIGURE 10, traveling clockwise around the cell on the right and counterclockwise around the cell on the left.

Let's call a walk *closed* if its terminal point coincides with its initial point. The shadow of a closed walk in Hoboken need not be closed in Manhattan; a walk that encircles a single block in Hoboken is an example of this. However, we do have the following crucial fact:

CLAIM 1. *If a region in Hoboken can be tiled by skew tetrominoes, then any closed walk in Hoboken that makes a complete circuit of the boundary of that region is shadowed by a closed path in Manhattan.*

Proof. We use induction on the area of the region, necessarily a multiple of 4. If the tileable region is just a single skew tetromino, then we are in the situation of FIGURE 10. To see that this picture is the only one we need to examine for the base case of our induction, note first that if you had chosen a different starting point for your walk in Manhattan, your walk would still have been closed, since Manhattan has symmetries (translations, rotations, and reflections) carrying any vertex to any other and respecting the pattern of arrows. Neither would the situation be affected if I had chosen a different starting point for my walk along the boundary of the skew tetromino, for that would be tantamount to having you start your walk at a different point in Manhattan. Finally, we have dealt with only one of the four orientations a skew tetromino can have, but the symmetries of Hoboken and Manhattan imply that what works for one orientation must work for the other three.

Now suppose we've proved the claim for all tileable regions with area less than $4n$. Consider a region R with area exactly $4n$ that comes equipped with a particular tiling. To prove that the boundary of R (a closed curve in Hoboken) is shadowed by a closed curve in Manhattan, I am going to modify my itinerary a bit, as suggested by FIGURE 11. Specifically, I am going to travel from a starting point p on the boundary of R to another point q on the boundary of R; then travel from q to p in the interior of R, traveling *only* on the boundaries of tiles in my tiling of R; then return from p to q by the same interior route; and finally travel in the same direction as before (clockwise or counterclockwise) from q to p along the boundary of R, completing my tour. The polygonal arc pq divides R into two regions, which we will call A and B.

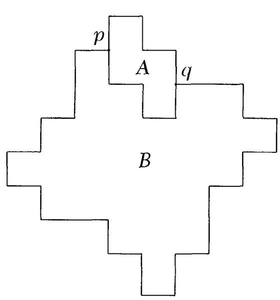

FIGURE 11
The induction step.

Since A and B each have area less than $4n$, and since each is tileable by skew tetrominoes, the boundary of each of them is shadowed by a closed path in the shadow-grid. Thus, the modified itinerary that I described above (from p to q to p to q to p) is shadowed by a closed path. However, the second and third legs of this journey (from q to p to q in the interior of R) are "inverses" of each other. It follows that as your walk in Manhattan shadows my (modified) walk in Hoboken, the second and third legs of your journey will also be inverses of each other, and the two together will leave you right where you were at the end of the first leg, when I was first at q. Hence, excising this pointless detour from my journey (and from yours), we see that the shadow of a path encircling R in Hoboken is also a closed path in Manhattan. This verifies the induction and completes the proof. ∎

Signed Area and the Main Theorem

We can strengthen Claim 1 using the notion of the signed area "enclosed" by a shadow path. To define this, we must first define the winding number of a finite closed curve around a point not on the curve. If we draw a ray emanating from the point, the winding number is simply the number of times that the curve crosses from one side of the ray to the other in the positive direction (counterclockwise) minus the number of times that the curve crosses from one side of the ray to the other in the negative direction (clockwise). If the closed curve consists of grid-edges, then the winding number of the curve around a point is the same for any other point in the same cell. We call this the winding number of the curve around the cell. For all but finitely many cells, the winding number of the curve around the cell is zero. Thus, we can speak of the sum of the winding numbers of the curve around all the cells; this is what I mean by the signed area enclosed by a curve. The reader should check that when the closed curve is simple, the signed area as I have defined it is equal to plus or minus the area enclosed by the curve as defined in the usual sense (positive if the curve winds counterclockwise, negative if the curve winds clockwise).

If the region R in Hoboken has a boundary whose shadow in Manhattan is a closed curve, we define the (signed) *shadow-area* of the region R as the signed area enclosed by the shadow of its boundary. We now show that a region that can be tiled by skew tetrominoes must have shadow-area equal to zero.

CLAIM 2. *If a region in Hoboken can be tiled by skew tetrominoes, then any closed walk in Hoboken that makes a complete circuit of the boundary of that region is shadowed by a closed path in Manhattan that encloses signed area* 0.

Proof. Again we use induction. The same pictures that worked before still work; we just have to examine them in a different frame of mind. In FIGURE 10, we need to notice that the shadow-curve has winding number $+1$ around one cell, -1 around another cell, and 0 around every other cell, giving it a total signed area of 0. As for the induction in FIGURE 11, we need to observe that for every place where the shadow of the internal path from q to p crosses a ray, the shadow of the return path from p to q crosses the same ray in the opposite direction. Thus the shadows of the second and third legs of your trip in Manhattan make canceling contributions to the winding number around any particular cell, and thus make canceling contributions to the signed area of the complete path. Excising the detour, we obtain the desired induction. ∎

You (the reader, not the walker) are now in a position where you can convince yourself of the untileability of many specific regions, including, as it happens, Aztec diamonds of any order. Specifically, take the boundary of your region in the Hoboken grid and shadow it in the Manhattan grid; if the resulting curve is not closed, or if it is closed but encloses non-zero signed area, then your region cannot be tiled by skew tetrominoes.

Satori

If you want to convince someone else that the region cannot be tiled, using the method as we have discussed it so far, then that person has to do essentially everything that you did. That is, he or she must carefully shadow the entire boundary and (if the shadow-path closes) work out the signed area of what may be a rather crazy self-intersecting closed curve. This is certainly a better way for you to convince people that a tiling problem is unsolvable than forcing them to read through a three-inch stack of coffee-stained sheets of graph paper in which all possibilities are tried and eliminated. But there is a still better way, in which a region R can be marked up in such a manner that, in the case where the boundary of R is shadowed by a closed curve, the proof-checker can see at a glance how much signed area is enclosed by the shadow-curve.

Recall that a square tetromino is a 2-by-2 square, as shown on the far right in FIGURE 1. The shadow of the boundary of a square tetromino in Hoboken is the boundary of a square tetromino in Manhattan, but the orientation may switch, giving it signed area $+4$ or -4. We need to nail down the sign exactly. Let's choose a particular corner in Hoboken and call it the Hoboken origin; we will say that another corner in Hoboken is *even* if it can be reached from the origin in an even number of steps (lengths of a city block) and *odd* if it can be reached from the origin in an odd number of steps. Meanwhile, let's find an intersection in Manhattan of "Hoboken type" (i.e., where the street goes east and the avenue goes north), and let that be the Manhattan origin. We define evenness and oddness of Manhattan intersections in an analogous way. If we assume that you and I begin our walks at vertices of the same parity, then it follows that you and I will be at vertices of the same parity as one another forever afterwards, since every block we walk changes the parity of your location and mine from odd to even or from even to odd.

Call a square tetromino in either grid *even* if its corners are at even vertices and *odd* otherwise. Consider the boundary of such a tetromino in Hoboken, traversed in the counterclockwise direction. The shadow of this path is also the boundary of a square tetromino, and with a little doodling you can check that the shadow path in Manhattan encircles signed area $+4$ or -4 according to whether the original square tetromino was even or odd.

We can now state our main result.

MAIN THEOREM. *Suppose a simply-connected region in the plane can be tiled by a mixture of skew tetrominoes and square tetrominoes. Then the number of even square tetrominoes minus the number of odd square tetrominoes does not depend on what tiling one chooses; i.e., it is an* invariant. *In particular, if the difference is non-zero for one such tiling, then the region cannot be tiled by skew tetrominoes alone.*

Proof. The difference in question, when multiplied by 4, is just the signed area enclosed by the shadow of the boundary of the region to be tiled, because each tetromino is shadowed by a path enclosing signed area $+4$, -4, or 0 according to whether it is an even square tetromino, an odd square tetromino, or a skew tetromino. The justification is the same as in the proof of Claim 2, namely, the additivity of signed area. ■

Now look back at FIGURE 7. The Aztec diamond of order 7 has been decomposed into a number of skew tetrominoes along with four square tetrominoes. Because all four square tetrominoes are even, the invariant has value $+4$. It follows that the region cannot be tiled by skew tetrominoes alone. In this sense, FIGURE 7 can be a proof-mandala for the impossibility of tiling the Aztec diamond of order 7 by skew tetrominoes, once the mind has absorbed the main theorem.

The mandala has even more to teach the receptive spirit. Notice that the Aztec diamond of order 7 has an Aztec diamond of order 6 sitting inside it, fringed above by skew tetrominoes and a single square tetromino. This order-6 diamond in turn contains of an order-5 diamond fringed above by skew tetrominoes. And so on. The mandala shows a clear iterative pattern for reducing an Aztec diamond of order $2k$ to an Aztec diamond of order $2k-1$ plus some skew tetrominoes, and for reducing an Aztec diamond of order $2k+1$ to an Aztec diamond of order $2k$ plus some skew tetrominoes and a single square tetromino, in such a way that the square tetrominoes all have the same parity. Thus, for general n, any tiling of the Aztec diamond of order n by skew tetrominoes and square tetrominoes must have an excess of exactly $\left\lfloor \frac{n+1}{2} \right\rfloor$ square tetrominoes of one particular parity. In particular, for $n \geq 1$, there can be no tiling of the Aztec diamond of order n by skew tetrominoes alone.

It would be interesting to know of a different way to prove the main theorem. One possible approach would be to mimic Donald West's proof **[9]** of Conway's triangle-tiling theorems, and show that every tiling of a simply-connected plane region by skew tetrominoes and square tetrominoes can be obtained from every other such tiling by means of a small repertoire of "local moves," each of which preserves the difference between the number of even square tetrominoes and odd square tetrominoes. Readers might experiment with such local moves and see if they can come up with a demonstrably complete set. My own guess is that a complete finite set of local moves does exist.

Extensions

As was mentioned earlier, the shadow-path method can be used to show that many particular regions R, and not just Aztec diamonds, cannot be tiled by skew tetrominoes. In fact, if one chooses R "at random," then it is likely that the shadow of its boundary will not be a closed path, or if it is closed, that it will not enclose signed area 0. Even if it does enclose signed area 0, a slight refinement of our approach may permit us to rule out the existence of a tiling. For, observe that the shadow of the boundary of a skew tetromino winds clockwise around one cell in the shadow-grid and counterclockwise around another cell *of the same color*, relative to the coloring shown in FIGURE 12. Thus, we can talk about (signed) A-area, B-area, C-area, and D-area, whose sum will be the signed area enclosed by a curve. In order for R to be tileable by skew tetrominoes, the shadow of the boundary of R must enclose A-area, B-area, C-area, and D-area all equal to 0.

A	B	A	B
C	D	C	D
A	B	A	B
C	D	C	D

FIGURE 12
Coloring the grid-squares.

This strengthened version of the main theorem goes a long way toward closing the gap between demonstrably tileable regions and demonstrably untileable regions. However, the gap is not altogether shut. For instance, the region of area 8 shown in FIGURE 13 is not tileable by skew tetrominoes, despite the fact that the main theorem (even in its strengthened form) does not tell us this. It would be valuable to know of an efficient algorithm that would close the gap completely, by deciding whether a given region is or is not tileable by skew tetrominoes.

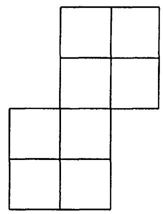

FIGURE 13
An untileable region.

It is also interesting to change the game and weaken the notion of "tileability" so that the preceding necessary condition becomes sufficient as well. We define a *tile homotopy* of a path in the grid as a process of perturbing the path by "pulling it through tiles." More precisely, an elementary homotopy between two closed grid-paths replaces a part of the path (call it P) joining two vertices p, q by another grid-path P' joining the same two vertices, such that P and P' together form the boundary of a tile. FIGURE 14 shows a series of elementary tile homotopies between the path shown in FIGURE 13 and the trivial loop; first the boundary is pulled outward by adding a skew tetromino on the outside and pulling the boundary through the new tile; then the path is pulled inward, using a tiling of the enlarged region by skew tetrominoes. It can be shown using combinatorial group theory that a closed path in Hoboken is tile-homotopic to the trivial loop if and only if its shadow in Manhattan is closed and encircles A-area, B-area, C-area, and D-area all equal to 0.

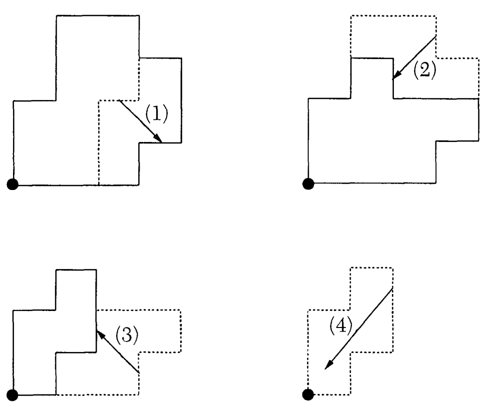

FIGURE 14
Tile homotopy.

The shadow-path method can be used to prove that in any partial tiling of the Aztec diamond of order n, the diameter of the untiled portion is bounded below by a constant times $\sqrt{n}$. For, one can find a small rectangle that covers the untiled portion and approximate the boundary of this rectangle by a closed loop L that travels only along the boundaries of tiles but is as direct as possible subject to that constraint. Since L is tile-homotopic to the boundary of the Aztec diamond, whose shadow encloses signed area roughly $2n$, the shadow of L must have length at least $c\sqrt{n}$ for some constant c. Hence L itself must have length at least $c\sqrt{n}$, implying that the untiled portion of the Aztec diamond has large diameter. It is hard to imagine a proof of such a result by means of the methods that predated Conway's work.

On the other hand, Aaron Meyerowitz has shown that there can be no analogous bound on the area of the untiled portion, by pointing out that for any $n \geq 1$ it is possible to tile the Aztec diamond of order n by $\frac{n(n+1)}{2} - 1$ skew tetrominoes, a single L-shaped tromino, and a single monomino. Specifically, we can tile the Aztec diamond of order n minus a 2-by-$2\left\lfloor \frac{n+1}{2} \right\rfloor$ rectangle that butts against a corner of the diamond, as shown in FIGURE 15. This rectangle can then be tiled by skew tetrominoes leaving only a tromino and a monomino unaccounted for. Note, however, that the tromino and the monomino are far apart, as the preceding lower bound on the diameter of the untiled portion requires.

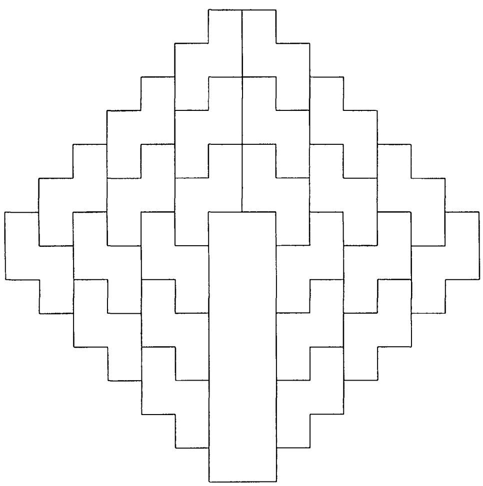

FIGURE 15
Variant proof-mandala.

One disadvantage of the elementary approach to Conway's invariants that has been adopted here is that the reader may be left feeling convinced but mystified: How might anyone dream up the traffic patterns of Hoboken and Manhattan that proved so useful here? And how can the arguments used here for skew tetromino tilings be generalized to handle other tiling problems? The answer lies in the notion of the tile homotopy group; the interested reader should consult [1]. A major idea in the Conway-Lagarias paper is to regard graphs like FIGURES 8 and 9 as Cayley graphs of groups, as I have implicitly done in the way I labeled and oriented the edges. One satisfying feature of the tile homotopy viewpoint is that coloring arguments of the sort considered earlier turn out to be a special case of Conway's method. Specifically, coloring arguments are associated with abelian homomorphic images of the tile homotopy group.

Thurston's follow-up paper **[8]** presents a more geometrical way of looking at tile homotopy that suggests that Cayley graphs of groups are not at the heart of the method. Given a collection of subsets of the plane (to be viewed as the set of all allowed locations of tiles), Thurston invites us to create a topological space by taking the disjoint union of all those subsets (imagined if you like as floating above the plane) and identifying two points on the boundaries of two such tile-regions if they lie above the same point in the plane. Assuming that the tiles are all simply-connected, it is easy to see that the boundary of any tileable region corresponds to a path in Thurston's space that is homotopic to the trivial loop. Thus we obtain a necessary condition for tileability from homotopy considerations, though actually exploiting this connection may be difficult in practice without recourse to Cayley-graph tricks. It turns out that the idea of boundary invariants can be extended beyond the realm of tilings, using Thurston's more geometrical approach; details appear in **[4]**.

I will end with an accessible puzzle that has a positive solution. The mandala teaches us that when $n = 2k^2 - 1$, the Aztec diamond of order n can be tiled by skew tetrominoes and k^2 square tetrominoes (all having the same parity). FIGURE 5 shows us that in the case $k = 3$, we can arrange things so that these k^2 square tetrominoes are all at the center of the diamond, forming a $2k$-by-$2k$ square. Can you find a way to do this for all k?

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Formulae, Algorithms, and Quartic Extrema

D. J. JEFFREY
University of Western Ontario
London, Ontario, N6A 5B7 Canada

Introduction

The reader probably thinks that finding the turning points of a quartic polynomial is a "solved problem," so the first thing to establish is the possibility of improving on the standard method, which is taken to be the calculus one. Thus, given a real polynomial

$$P(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0, \tag{1}$$

whose turning points are required, one sets the derivative to zero and solves the cubic

$$P'(x) = 4a_4 x^3 + 3a_3 x^2 + 2a_2 x + a_1 = 0. \tag{2}$$

This will have 1 or 3 real roots; if there are 3 real roots, one must test each one separately to ascertain the nature of the turning point, and hence finally obtain the global extremum. The process of finding extrema has been automated by several computer algebra systems (CAS), to a greater or lesser extent.

So what is there to improve? If the coefficients are known numerically, then not very much, but suppose that some of the coefficients a_i are known only symbolically. Consider the ways in which mathematicians give solutions to problems containing symbolic parameters. The first way is to give a formula, meaning an explicit function of the parameters. This is the method fixed in the popular imagination: spies in espionage stories have always chased after *the formula*. For example, the infimum of a quadratic polynomial, a result used below, is given by the following formula. If $a_2 > 0$, then

$$\inf\left(a_2 x^2 + a_1 x + a_0\right) = a_0 - \frac{a_1^2}{4a_2}, \tag{3}$$

and, moreover, the value of x that gives this infimum is $x_f = \frac{1}{2}a_1/a_2$, another formula. We recall that this well-known result can be deduced without calculus by completing the square. Among CAS, Maple is able to return (3) through its `minimize` command, although it has no syntax for returning the position x_f.

In contrast to the solution of the quadratic problem, the solution of the quartic problem was given above in the form of a *procedure* or *algorithm*, not a formula.

THEOREM 1. *Formulae are better than algorithms.*

Proof. There has never been a story or film in which rival groups of spies chase, explode, and kill each other in order to gain possession of an algorithm. For a formula, on the other hand, they will "stop at nothing." Q.E.D.

Of course, many think that spy stories rely too much on formulas, but that is a different topic. Certainly computer algebra systems prefer formulae, and their syntax has to be stretched in order to accommodate algorithms. Maple does this when the command `minimize(`$P(x)$, x`)` returns

$$\frac{1}{16a_4}\left[\left(8a_4a_2-3a_3^2\right)X^2+\left(12a_4a_1-2a_3a_2\right)X-a_3a_1\right]+a_0,$$

with

$$X=\texttt{RootOf}\left(4a_4Z^3+3a_3Z^2+2a_2Z+a_1,Z\right).$$

The first argument of the Maple function `RootOf` is the equation to solve, and the second argument is the variable to solve for; then X can be any one of the roots of the given equation. Notice, in passing, that Maple has used the properties of X to reduce the quartic (1) to a quadratic expression.

We can now pose the problem to be considered: can the algorithm (1) and (2) be replaced by a formula, or at least some formulae. Not only is the answer *yes*, but you will not have to brave a single scorpion to learn it (just perhaps a few cobwebs).

Formula One

As an opening, we can reduce the number of unknowns we have to face by taking out a few coefficients. Dividing through by a_4 makes the polynomial in (1) *monic*, and one term can be disposed of by shifting to the variable $y=x+\frac{1}{4}a_3/a_4$. Actually, a_3 and a_4 are like minor spies at the start of the movie: they were included so that you would be impressed by their swift elimination. Any constant terms simply raise or lower everything and can be added on at the end. Cast into formal language, these transformations become a lemma.

LEMMA 2. *Provided* $a_4>0$, *the infimum of a general quartic is given by*

$$\inf\left(a_4x^4+a_3x^3+a_2x^2+a_1x+a_0\right)=a_4\inf\left(y^4+3b_2y^2+2b_1y\right)+a_0+\beta_0,$$

where the factors 3 *and* 2 *simplify later formulae, and*

$$b_2=\frac{a_2}{3a_4}-\frac{a_3^2}{8a_4^2},\quad b_1=\frac{a_1}{2a_4}+\frac{a_3^3}{16a_4^3}-\frac{a_2a_3}{4a_4^2},\quad \beta_0=\frac{a_2a_3^2}{16a_4^2}-\frac{a_1a_3}{4a_4}-\frac{3a_3^4}{256a_4^3}.$$

The derivation of the minimum of the reduced polynomial $y^4+3b_2y^2+2b_1y$ has to treat the cases $b_1=0$ and $b_1\neq 0$ separately. So starting with the more general case $b_1\neq 0$, we can plant our first formula in a theorem.

THEOREM 3. *If the coefficient* $b_1\neq 0$, *the quartic polynomial*

$$P_4(y)=y^4+3b_2y^2+2b_1y \tag{4}$$

has an infimum on the real line given by

$$\inf P_4=M(b_1,b_2)=-\tfrac{3}{4}(k_f-b_2)(k_f-3b_2), \tag{5}$$

where

$$k_f=s^{1/3}+b_2^2s^{-1/3}+b_2 \tag{6}$$

$$s=b_1^2+b_2^3+\sqrt{b_1^4+2b_1^2b_2^3} \tag{7}$$

and $s^{1/3}$ *and* $s^{-1/3}$ *are always interpreted as the principal values of the powers. Moreover, the infimum of* P_4 *is located at* $y = y_f = -b_1/k_f$.

Proof. The idea is to break $P_4(y)$ into two pieces in such a way that each piece can be minimized easily. Then one uses the fact that any two polynomials $f(y)$ and $g(y)$, both bounded below, obey

$$\inf(f(y)+g(y)) \geq \inf f(y) + \inf g(y),$$

with equality holding when the same value of y minimizes both f and g. When we split P_4 into two polynomials, denoted $P_4^{(1)}$ and $P_4^{(2)}$, we also introduce a parameter k, which is assumed to satisfy $k > 0$.

$$P_4 = \left[y^4 + (3b_2 - k)y^2\right] + \left[ky^2 + 2b_1 y\right] = P_4^{(1)} + P_4^{(2)}.$$

So long as k is positive, the parabola $P_4^{(2)}$ has a minimum. FIGURE 1 illustrates the splitting of P_4. The turning points of the two parts can now be found without

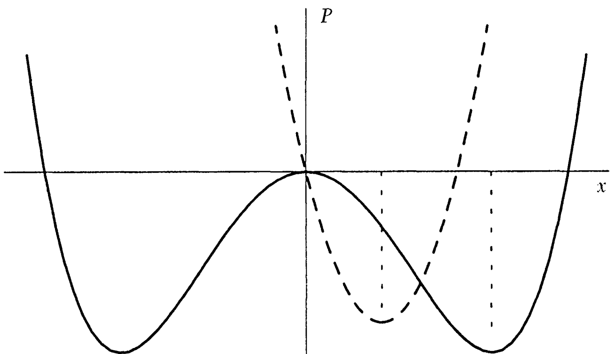

FIGURE 1
The splitting of the quartic into two parts. The fourth-degree part (solid line) is symmetric. The positions of the minima vary with k. As k decreases, the parabola (dashed) moves out. For the value of k that makes the two minima coincide, the minimum of the original quartic is obtained.

calculus. Completing the square can be used to rewrite $P_4^{(1)}$ as

$$P_4^{(1)} = y^4 - (k-3b_2)y^2 = \left[y^2 - \tfrac{1}{2}(k-3b_2)\right]^2 - \tfrac{1}{4}(k-3b_2)^2.$$

If it is additionally assumed that $k - 3b_2 > 0$, then $P_4^{(1)}$ has the minimum $-\frac{1}{4}(k - 3b_2)^2$ at the points where $y^2 = \frac{1}{2}(k-3b_2)$. The infimum of $P_4^{(2)}$ is $-b_1^2/k$, by (3), and therefore

$$\inf(P_4) \geq -b_1^2/k - \tfrac{1}{4}(k-3b_2)^2. \tag{8}$$

If a value of k can be found that forces the two points where $P_4^{(1)}$ and $P_4^{(2)}$ have minima to coincide, then we have also located the infimum of P_4. Of course, any such value for k will also have to satisfy our accumulated assumptions $k > 0$ and $k > 3b_2$. The turning point for $P_4^{(2)}$, being at $y = -b_1/k$, moves closer to the origin as k increases, while that of $P_4^{(1)}$, being at $y^2 = \frac{1}{2}(k-3b_2)$ moves away. So they will coincide when k takes a value such that

$$\tfrac{1}{2}(k-3b_2) = (-b_1/k)^2.$$

This is equivalent to the cubic equation,

$$k^3 - 3b_2k^2 - 2b_1^2 = 0. \tag{9}$$

I shall call $C(k) = k^3 - 3b_2k^2 - 2b_1^2$ the *auxiliary cubic*. Another way to get this equation is to use calculus to maximize the right side of (8) directly. The cubic equation (9) has a unique positive solution. The intermediate value theorem could be used to show this analytically, but the equation plays a central role in what follows, so it is better to understand its properties graphically. FIGURES 2–4 show plots of $C(k)$ for the different ranges of the parameters. The twists in the plots can be understood by noticing that $C(0) = C(3b_2) = -2b_1^2$ and $C'(0) = C'(2b_2) = 0$. Also $C(2b_2) = C(-b_2) = -2b_1^2 - 4b_2^3 = -2D$, after introducing the abbreviation $D = b_1^2 + 2b_2^3$.

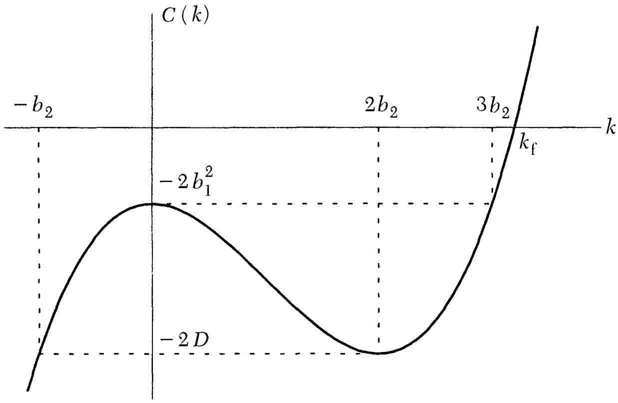

FIGURE 2
A graph of the auxiliary cubic for $b_2 > 0$. It can be seen that the root k_f is positive and greater than $3b_2$ as required. The quantity $D = b_1^2 + 2b_2^3$ is positive.

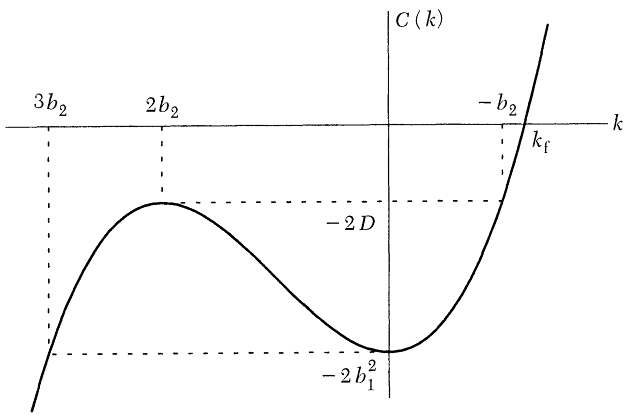

FIGURE 3
A graph of the auxiliary cubic for $b_2 < 0$. The quantity $D = b_1^2 + 2b_2^3$ is positive, so there is only one real root.

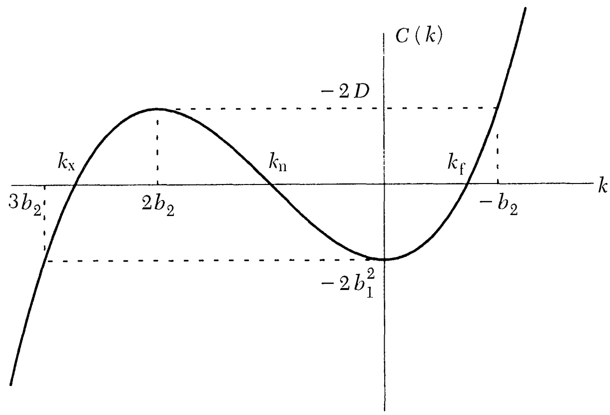

FIGURE 4
A graph of the auxiliary cubic for $b_2 < 0$. The quantity $D = b_1^2 + 2b_2^3$ is negative, so there are 3 real roots. They are marked on the figure as k_x, k_n and k_f.

Denote the positive solution of (9) by k_f. Since $k_f > 3b_2$, the minimum of (4) is obtained from (8) as

$$\inf(P_4) = -b_1^2/k_f - \tfrac{1}{4}(k_f - 3b_2)^2.$$

The first term of this expression is better transformed so that b_1 does not appear explicitly, for reasons that will be given below. The transformation is made by rewriting (9) in the form

$$\tfrac{1}{2}k^2 - \tfrac{3}{2}b_2 k = b_1^2/k, \tag{10}$$

and hence (5) is obtained.

It remains to find an explicit formula for k_f. The expression (6) is a standard solution of (9), but the fact that it gives the positive solution of (9) must be verified. Rewrite (7), introducing D, as $s = b_1^2 + b_2^3 + \sqrt{b_1^2 D}$. First, consider the case $b_2 \geq 0$ and $D \geq 0$; all terms in (6) are real and positive. Second, consider $b_2 < 0$ and $D \geq 0$, meaning $b_1^2 \geq -2b_2^3$. Then $s > -2b_2^3 + b_2^3 + \sqrt{b_1^2 D} > -b_2^3$, and therefore $s^{1/3} > -b_2 > 0$, and $k_f > 0$. Finally, if $b_2 < 0$ and $D < 0$, then s will be complex, explicitly $s = b_1^2 + b_2^3 + i\sqrt{-b_1^2 D}$. Squaring and adding the real and imaginary parts shows $|s|^2 = b_2^6$, and so in polar form $s = -b_2^3 e^{i\theta}$ with $0 < \theta < \pi$, since the imaginary part is positive and therefore in the upper half-plane. Then

$$s^{1/3} + b_2^2 s^{-1/3} = -b_2 e^{i\theta/3} - b_2 e^{-i\theta/3} = -2b_2 \cos\tfrac{1}{3}\theta, \tag{11}$$

and since $2\cos\frac{1}{3}\theta > 1$, the value of k_f is real and positive.

The special case $b_1 = 0$ must now be taken up, but it is an easy one because there are only two terms in the polynomial.

THEOREM 4. *For the case $b_1 = 0$, the polynomial $P_4(y) = y^4 + 3b_2 y^2$ has the infimum*

$$\inf P_4 = -\tfrac{9}{4}\min(0, b_2)^2,$$

at the points $y^2 = -\frac{3}{2}\min(0, b_2)$.

Proof. If $b_2 \geq 0$, then clearly the minimum is 0 when $y = 0$. If $b_2 < 0$, then completing the square can be used again to give the minimum as $-\frac{9}{4}b_2^2$ at the points $y^2 = -\frac{3}{2}b_2$. The theorem uses the minimum function to combine these cases.

Now there is an interesting development. One takes formula (5) for $M(b_1, b_2)$, forgets that it was derived for $b_1 \neq 0$, and substitutes $b_1 = 0$. For the case $b_2 > 0$, one computes $k_f = 3b_2$ and $M(0, b_2) = 0$. For $b_2 < 0$, equation (11) can be reused with $\theta = \pi$, making $s = -b_2$ and $k_f = 0$. Then $M(0, b_2) = -\frac{9}{4}b_2^2$. Thus for these cases, M continues to give the correct result. This is because of the transformation (10). For $b_2 = 0$, k_f contains a term $0/0$ and this prevents a simple substitution from obtaining $M(0,0) = 0$, or in other words, $M(b_1, b_2)$ has a removeable singularity at $b_1 = b_2 = 0$. Unfortunately, this trick cannot be repeated for the position of the infimum y_f, and that has to remain having a piecewise definition.

$$y_f(b_1, b_2) = \begin{cases} -b_1/k_f, & b_1 \neq 0, \\ \sqrt{-\min(0, 3b_2/2)}, & \text{otherwise.} \end{cases}$$

The positive root has been arbitrarily chosen for definiteness in the case $b_1 = 0$.

A Secondary Formula

A quartic polynomial can have 3 turning points, corresponding to (2) having 3 real roots. It is also possible for (9) to have 3 real roots. Is there a connection? At first sight, it seems not, because k was assumed to be positive, and there is only one positive solution of (9). In spite of this doubt, negative values of k do indeed give the other turning points. First we give a formula for the secondary minimum.

THEOREM 5. *If the coefficient* $b_1 \neq 0$, *and* $D = b_1^2 + 2b_2^3 < 0$, *the quartic polynomial* $P_4(y)$ *defined in* (4) *has a secondary minimum* $N(b_1, b_2)$ *equal to*

$$N(b_1, b_2) = 3b_2 k_n - \tfrac{3}{4}k_n^2 - \tfrac{9}{4}b_2^2,$$

where

$$k_n = s^{1/3}e^{-2\pi i/3} + b_2^2 s^{-1/3} e^{2\pi i/3} + b_2,$$

and s is unchanged from equation (7). *Moreover, the secondary minimum is located at* $y = y_n = -b_1/k_n$.

Proof. The only quantity that is different from Theorem 3 is k_n, which is a different solution of (9). FIGURE 4 illustrates that it is the root satisfying $2b_2 < k_n < 0$, as we now show. For the given range of parameters, $s = -b_2^3 e^{i\theta}$, where $0 < \theta < \pi$, as in (11). Therefore $k_n = b_2(1 - 2\cos\phi)$, with $-\frac{2}{3}\pi < \phi \leq -\frac{1}{3}\pi$. Even if the parameter k is negative, it is still true that $P_4 = P_4^{(1)} + P_4^{(2)}$, and the derivatives of P_4 can be calculated by adding those of $P_4^{(1)}$and $P_4^{(2)}$. The derivatives of P_4 at $y = y_n$ are thus computed to be

$$\frac{dP_4}{dy}(y_n) = 0 \quad \text{and} \quad \frac{d^2P_4}{dy^2}(y_n) = 6k_n - 12b_2 \geq 0.$$

Therefore, the point y_n is a local minimum, but not the infimum, which corresponds to a positive value of k.

By now, it is clear that the third root of $C(k)$ gives the relative maximum between the two minima.

THEOREM 6. *With the notation already defined, P_4 has a relative maximum when $D<0$ and k_x is the root of $C(k)$ satisfying $3b_2<k_x<2b_2$. The formula for k_x is*

$$k_x = s^{1/3}e^{2\pi i/3} + b_2^2 s^{-1/3}e^{-2\pi i/3} + b_2 .$$

Properties of the Solutions

In spy stories, everyone seems to think that possessing the formula is all that is required—a bit like a weak student facing a mathematics exam. However, one must be able to use it. The formulae just derived can be used to show that the turning points of P_4 have some interesting properties. For example, the sign of b_1 is all that decides the side of the origin on which the infimum lies; if there is a secondary minimum, then both it and the local maximum are always on the same side of the origin, and that is the opposite side from the infimum; the infimum is always further away from the origin than the secondary minimum.

After several pages of algebra, it is always comforting to try a few numerical examples and see that everything works out. No one wants a repeat of the last scenes of *The Maltese Falcon*. In addition, the examples here carry some useful lessons of their own. So consider y^4-14y^2-24y, which of course has been carefully rigged to have integer turning points. Substituting $b_2=-14/3$ and $b_1=-12$ into (6) and asking Maple to simplify the result gives

$$k_f = \frac{2\left(143+180\sqrt{3}\,i\right)^{2/3} + 98 - 14\left(143+180\sqrt{3}\,i\right)^{1/3}}{3\left(143+180\sqrt{3}\,i\right)^{1/3}} .$$

All computer systems can approximate this to 4.0000000, but none can automatically simplify it to the exact number 4. This is an unavoidable difficulty associated with solving a cubic using the standard formulae. The simplification $(143+180\sqrt{3}\,i)^{1/3} = \frac{1}{2}(13+3\sqrt{3}\,i)$, which is needed to obtain the exact result, is not implemented in any present computer system; perhaps not many humans would make the simplification spontaneously either. Of course most of the time, no simplification is possible. In any event, the infimum is at $x_f=3$, and equals -117. In the same way, and with the same difficulties, $k_n=-6$ and $k_x=-12$.

Simply using the formula M given in (5) to compute a numerical minimum is not a very interesting application. A more challenging question is to find the values of p that make the polynomial x^4+3px^2+2x+2 positive for all x. This type of problem is a simple example of quantifier elimination **[1]**. The condition is simply $M(1,p)+2>0$, which becomes a long messy inequality when written out explicitly. Plotting the expression numerically shows that the answer is $p>-1/3$, but an analytic proof is a real challenge.

The final example does not aspire to present a general method for a class of problems, but the following challenge arose at the time of writing this paper. Given the points (x_i, y_i) equal to $(0,4)$, $(1,2)$, $(3,1)$, $(4,2)$, $(6,5)$, find a convex polynomial that passes through them. The Lagrange interpolating polynomial is a quartic:

$$y_L = \sum_{i=1}^{5} y_i \prod_{j\neq i} \frac{x-x_j}{x_i-x_j} = -\tfrac{1}{90}x^4 + \tfrac{4}{45}x^3 + \tfrac{13}{45}x^2 - \tfrac{71}{30}x + 4.$$

This does not satisfy $y''>0$ everywhere. Therefore we investigate whether a sixth-degree polynomial can be found. For unknown coefficients a and b, we write

$$y_S = y_L + (ax+b)\prod_{i=1}^{5}(x-x_i).$$

For all a and b, this passes through the given points. Two derivatives of this give a quartic inequality $y_S'' > 0$, which will be satisfied if $\inf y_S'' > 0$. This reduces to an inequality in the two variables a and b after using (5). Plotting contours shows that a region exists that satisfies the constraint, and in particular it includes the rectangle $1/2000 < a \le 1/1000$, $0 < b \le 1/1000$.

A Computer Epilogue

Just when you think it is time to roll the references, a familiar character reappears: the computer. The introduction mentioned that many CAS have routines to minimize functions, so the programming aspects of Theorem 3 are of interest. It was seen above that if (6) is used with numerical coefficients, the system might obtain a result that is correct, but not in the simplest form. In a similar way, if the evaluation of (6) generates intermediate quantities that are complex, a small nonzero imaginary part can appear in the final result because of rounding errors.

An alternative to the explicit formula (6) is to use a token such as the `RootOf` offered in Maple V Release 4. To the description above, we can add that `RootOf` accepts a third argument, in the form of an interval that brackets the required root. The interval could be specified using (6), of course, but it is better to surrender precision to gain algebraic simplicity. Now $k_f \to 3b_2$ as $b_2 \to \infty$, and $k_f \to (2b_1^2)^{1/3}$ as $b_2 \to 0$,
and $k_f \to \sqrt{-2b_1^2/3b_2}$ as $b_2 \to -\infty$. An estimate that takes these limits into account, while staying with integral powers is $k_a = 1 + 3|b_2| + \frac{2}{3}b_1^2$, which is an upper bound on k_f because $C(k_a) > 0$. Thus, the expression (6) can be replaced by

$$k_f = \texttt{RootOf}\big(k^3 - 3b_2k^2 - 2b_1^2, k, 0\,..\,1 + 3|b_2| + \tfrac{2}{3}b_1^2\big),$$

where some artistic license has been taken with Maple's input language. Similar constructions exist in other systems. The advantages of this approach are that the system has the possibility of obtaining the best representation of the root directly, and that the case $b_1 = b_2 = 0$ is no longer a removable singularity. The disadvantages are that the representation is unfamiliar, and it may not allow the further analysis that is possible with the explicit form; also the simplification routines existing for this type of construction are not yet at all strong.

A final comment repeats what has been achieved from a slightly different perspective. The paper opened with a cubic equation (2) whose roots gave turning points. The main theorem replaced this with a different cubic. One cannot avoid solving a cubic sooner or later, but the auxiliary cubic in (9) has the advantage that one knows in advance which root to select and where it will be.

Acknowledgment. *The last example came from a discussion with S. Watt, J. Grimm, and A. Galligo; in particular, I thank J. Grimm for showing me some related results obtained from a more general point of view. This article was written while on leave at INRIA, Sophia Antipolis, France, and the hospitality of S.M. Watt is gratefully acknowledged.*

One of the referees proposed a counterexample to Theorem 1: unfortunately my agents have been unable to locate the movie *Sneakers* to verify this intelligence report.

REFERENCE

1. Lazard, D., Quantifier elimination: optimal solution for two classical problems, *J. Symbolic Comp.* 5 (1988), 261–266.

Markov Chains and the *RISK* Board Game

BARIŞ TAN
Graduate School of Business, Koç University
Çayir Cad. No. 5 Istinye 80860 Istanbul, Turkey

1. Introduction

Markov chains have been applied in a wide variety of areas, including production, linguistics, finance, marketing, computer science, and signal processing. The first applications of Andrei Andreyevich Markov (1856–1922) were in modeling the progress of consonants and vowels in the writings of Pushkin and Aksakov [**2**].

In this study, we answer two questions of interest to *RISK* players by using Markov chains. At each turn, a player must decide whether or not to attack a territory. The first question is the following: *If you attack a territory with your armies, what is the probability that you will capture this territory?* Of course, the probability that you will capture a territory could be high while the expected loss is also high. Therefore, our second question is the following: *If you engage in a war, how many armies should you expect to lose depending on the number of armies your opponent has on that territory?*

Markov chains have also been applied to other board games [**1**], [**2**]. Ash and Bishop [**2**] calculated the steady state probability of a player landing on any Monopoly square under the assumption that each Monopoly player who goes to Jail stays there until he or she rolls doubles or has spent three turns in Jail. This model leads to a very practical observation for people who play the game often: the Orange monopoly (Tennessee Ave., St. James Place, and New York Ave.) is the most profitable.

Games have been subject to numerous studies in applied mathematics, operations research, and economics. The major difference between mathematical modeling of games and mathematical modeling of *real* systems is in the approximation phase. A model builder approximates the real world, according to the intent of use of such a model, under some restrictive assumptions; then the model is analyzed with mathematical techniques. Thus the results obtained from the mathematical model are approximations to the real system. Games can be viewed as simplified versions of real life situations, so mathematical models of games can be developed under fewer restrictive assumptions. In our mathematical model, the results for a single war of the *RISK* board game are obtained under no restrictive assumptions. Therefore our results are exact for the game itself.

2. Rules of the Game

The *RISK* game has a world map separated into 42 territories. Each player has a different number of tokens that are used as armies. Each token represents one army. The number of tokens is determined at each turn according to the territories each player occupies. The objective of a player in the game is either to conquer the world or to fulfill a task assigned at the beginning of the game. This task can be destroying a specific enemy or conquering a specific area.

At each turn, a player may attack a territory from a neighboring territory occupied by the attacker. An attacker must leave at least one army on the attacking territory, and at least one army must be used in an attack. Thus a territory may be attacked by a player who has at least two armies (one to keep on the territory and one to attack) on a neighbor territory. A player who decides to attack a territory declares war on the opponent and the two players engage in a war. Dice are used in the war; successive outcomes of the defender's and the attacker's dice determine the outcome of the war. An attacker may withdraw before the war ends. An attacker who does not withdraw either destroys all the defender armies in that territory and occupies that territory, or loses all the attacking armies and fails to conquer the territory.

The number of armies an attacker or a defender has determines the number of dice each player rolls. A defender rolls two dice with two or more armies, and one die with one army. An attacker rolls three dice with three or more attacking armies, rolls two dice with two, and rolls only one die with only one army. According to the number of dice each opponent rolls, there are six different cases, given in Table 1.

At each turn, after the dice have been rolled, each side's dice are placed in descending order and the two sets of dice are paired off. The attacker loses one army for each die that is less than or equal to the corresponding defender's die. The defender loses one army for each die that is less than the corresponding attacker's die. After the armies lost at that turn are taken away, the dice are rolled again. This continues until one side loses all of its armies. An example of a war is given below. This war took four turns. At the end of the fourth turn, all of the attacking armies are lost and the defender wins the war.

Turn #	Number of armies		Number of dice rolled		Outcome of the dice		Number of losses	
	attacker	defender	attacker	defender	attacker	defender	attacker	defender
1	4	3	3	2	5, 4, 3	6, 3	1	1
2	3	2	3	2	5, 5, 2	5, 5	2	0
3	1	2	1	2	6	4, 3	0	1
4	1	1	1	1	5	6	1	0
5	0	1						

3. A State-Space Model

Before we can determine the probability of conquering a territory and the expected losses of both sides, we need to develop a state-space model of a single war. Let A be

TABLE 1 The number of dice each side rolls according to the number of attacker and defender armies

Case	Number of Armies		Number of Dice Rolled	
	Attacker	Defender	Attacker	Defender
I	1	1	1	1
II	2	1	2	1
III	≥ 3	1	3	1
IV	1	≥ 2	1	2
V	2	≥ 2	2	2
VI	≥ 3	≥ 2	3	2

the number of attacking armies and D the number of defending armies. The state of the system at a given time is characterized by A and D. Let X_n be the state of the system at the beginning of the nth turn:

$$X_n = (a_n, d_n),\ 0 \le a_n \le A,\ 0 \le d_n \le D$$

where a_n and d_n are the number of attacking and defending armies, respectively. The initial state of the system is $X_0 = (A, D)$.

If one side loses all its armies then that side loses the war; i.e., if $X_m = (0, d_m)$ with $d_m > 0$, then the attacker has lost the war at the end of the previous turn. Similarly, if $X_m = (a_m, 0)$, with $a_m > 0$, then the defender has lost the war at the end of the previous turn.

If we know the number of armies each side has at a given turn, then we can calculate the probability that each side wins or loses the war without knowing the states of the system prior to that turn. In other words, this process has the Markov property:

$$\begin{aligned}P[X_{n+1} = (a_{n+1}, d_{n+1}) | X_n = (a_n, d_n), X_{n-1} = (a_{n-1}, d_{n-1}), \dots, X_0 = (A, D)] \\ = P[X_{n+1} = (a_{n+1}, d_{n+1}) | X_n = (a_n, d_n)].\end{aligned}$$

Thus $\{X_n\colon n = 0, 1, 2, \dots\}$ is a Markov chain, with state space $\{(a, d)\colon 0 \le a \le A, 0 \le d \le D\}$.

If the process starts at (A, D), it terminates either at $X_m = (0, d_m)$, with $d_m > 0$, or $X_m = (a_m, 0)$, with $a_m > 0$. In other words, one side either wins or loses. The states $(0, d_m)$, with $d_m > 0$, and $(a_m, 0)$, with $a_m > 0$ are called the absorbing states. The sum of the probabilities that the process terminates at $(0, d_m)$, $d_m > 0$ is the probability that the attacker loses (defender wins). Similarly, the sum of the probabilities that it terminates at $(a_m, 0)$, $a_m > 0$ is the probability that the attacker wins (defender loses).

If only the winning probabilities are of interest, then the states $(0, d_m)$, $d_m > 0$ can be lumped into a single state, say Z, denoting that the defender wins. Similarly the states $(a_m, 0)$, $a_m > 0$ can be lumped into a single state, say K, denoting that the attacker wins. Doing so reduces the dimensions of the state space. Since we are interested in the expected losses of each side, we have not lumped these states.

Because one side either wins or loses at the end of a war, state $(0, 0)$ cannot be reached from any other state. Therefore there are $A \cdot D + A + D$ states in the state space. Let the states of this system be ordered as $\{(1,1), (1,2), \dots, (1, D), (2,1), (2,2), \dots, (2, D), \dots, (A, 1), (A, 2), \dots, (A, D), (0,1), (0,2), \dots, (0, D), (1,0), (2,0), \dots, (A, 0)\}$, and let the states be indexed from 1 to $A \cdot D + A + D$. With this ordering, the first $A \cdot D$ states are transient and the remaining $A + D$ states are absorbing.

Let $\underline{P} = \{p_{ij}\}$ be the one-step state transition matrix; its elements p_{ij} denote the probability that the index of the state of the system at the beginning of the next turn is j given that the index of the state of the system at the beginning of this turn is i. We will find these probabilities in the next section. The matrix $\underline{P}$ has $A \cdot D + A + D$ rows and $A \cdot D + A + D$ columns, and has the following form:

$$\underline{P} = \begin{bmatrix} Q & R \\ \underline{0} & I \end{bmatrix}.$$

Here Q is an $A \cdot D \times A \cdot D$ matrix; its elements are the probabilities of transition only between the transient states for its elements. Similarly, R is an $A \cdot D \times (A + D)$

matrix whose elements are one-step transition probabilities from transient state to absorbing states, $\underline{0}$ is a $(A+D)\times A\cdot D$ matrix with all elements equal to zero, and I is the $(A+D)\times(A+D)$ identity matrix.

Let $f_{ij}^{(n)}$, $i=1,2,\ldots, A\cdot D$ and $j=1,2,\ldots, A+D$, be the probability that starting from a transient state with index i, the process enters an absorbing state with index $j+A\cdot D$ in n turns. Let $F^{(n)}$ be the matrix whose ijth element is $f_{ij}^{(n)}$. Note that if the process enters an absorbing state at the end of the nth turn, the transitions in the first $n-1$ turns must be among transient states and the transition at the nth turn must be from a transient state to an absorbing state. Therefore

$$F^{(n)}=Q^{n-1}R.$$

The process can terminate at the end of any turn, so the probabilities that the process eventually terminates at one of the absorbing states are calculated by summing $F^{(n)}$ from $n=0$ to infinity. That is,

$$F=\sum_{n=0}^{\infty}F^{(n)}=\sum_{n=0}^{\infty}Q^{n-1}R=(I-Q)^{-1}R, \tag{1}$$

where F is a $A\cdot D\times(A+D)$ matrix. The ith column of $F, F^{\langle i\rangle}$, contains the probabilities that the process eventually terminates at the absorbing state i, given the initial states (a,d), $0<a\le A$, $0<d\le D$, $i=1,2,3,\ldots, A+D$ for the states $(0,1),(0,2),\ldots,(0,D),(1,0),(2,0),\ldots,(A,0)$, respectively.

Since the defender wins the war if the process terminates at the states $(0,d)$, $0<d\le D$, the vector P_K whose elements are the winning probabilities of the defender given an initial state (a,d), $0<a\le A$, $0<d\le D$, is obtained by adding the first D columns of the matrix F:

$$P_K=\sum_{i=1}^{D}F^{\langle i\rangle}.$$

Similarly, the vector P_Z whose elements are the winning probabilities of the attacker given an initial state (a,d), $0<a\le A$, $0<d\le D$, is obtained by adding the last A columns of the matrix F:

$$P_Z=\sum_{i=D+1}^{A+D}F^{\langle i\rangle}.$$

Since each side either wins or loses, P_K+P_Z is a vector with elements equal to one.

Expected losses of defender and attacker are also of interest: the probability of winning for one side could be high while the expected loss is also high. For the attacker, this may change the decision of whether or not to attack a territory.

When the defender wins, the state of the system is $(0,d)$, $d>0$, where d is the number of remaining defender armies. Multiplying d by the probability that the process terminates in state $(0,d)$ and summing over all d with $0<d\le D$ gives the expected number of remaining defender armies. Expected remaining attacking armies can be determined in the same way.

Let E be the $(A+D)\times 2$ matrix defined as

$$E=\begin{bmatrix}1 & 2 & \cdot & D & 0 & 0 & \cdot & 0\\ 0 & 0 & \cdot & 0 & 1 & 2 & \cdot & A\end{bmatrix}^T,$$

and let ER be the $(A\cdot D)\times 2$ matrix whose first and second columns are the expected number of remaining defender and attacker armies, respectively, given an initial state (a, d), $0 < a \leq A$, $0 < d \leq D$. Clearly,

$$ER = F \cdot E.$$

The expected losses of a defender or an attacker are found by subtracting the number of expected remaining armies from D and A, respectively.

For more detailed information about analysis of Markov chains, the reader is referred to [**3**].

4. Determining the State Transition Probabilities

When two opponents engage in a war, an attacker rolls at most three dice and a defender at most two, as shown in Table 1. The state transition probabilities depend on the number of dice rolled. Before determining these probabilities, we consider the probability distributions of the maximum of one, two, and three dice and of the second largest number of two and three dice shown in Table 2.

TABLE 2 The probability distributions of the maximum of one, two, three dice and the second largest number of two and three dice

Number of dice		1	2	3	4	5	6
1	outcome of the die	1/6	1/6	1/6	1/6	1/6	1/6
2	maximum	1/36	3/36	5/36	7/36	9/36	11/36
	second largest	11/36	9/36	7/36	5/36	3/36	1/36
3	maximum	1/216	7/216	19/216	37/216	61/216	91/216
	second largest	16/216	40/216	52/216	52/216	40/216	16/216

With the information in Table 2, the state transition probabilities can be calculated for the six different cases given in Table 1. The maximum number of armies an attacker or a defender loses is 2. Therefore from a state (a, d), the possible transitions are to states $(a-2, d)$, $(a-1, d)$, $(a, d-1)$, $(a, d-2)$, and $(a-1, d-1)$, provided that $a \geq 2$ and $d \geq 2$.

As an example of the state transition probability calculations, consider case 6, where the attacker rolls three dice and the defender rolls two. Let Z_1 and Z_2 be the random variables denoting the outcomes of the defender's dice, $Z^{(1)}$ be the maximum and $Z^{(2)}$ be the second largest of Z_1 and Z_2. Similarly, let Y_1, Y_2 and Y_3 be the random variables denoting the outcomes of the attacker's dice, $Y^{(1)}$ be the maximum and $Y^{(2)}$ be the second largest of Y_1, Y_2 and Y_3. If $Y^{(1)} > Z^{(1)}$ and $Y^{(2)} > Z^{(2)}$ then the defender loses two armies. In the opposite case, i.e., if $Y^{(1)} \leq Z^{(1)}$ and $Y^{(2)} \leq Z^{(2)}$, the attacker loses two armies. In all the other cases, both attacker and defender lose one army. Thus the state transition probabilities from the state (a, d) (with $a \geq 3$ and $d \geq 2$) to states $(a-2, d)$, $(a-1, d-1)$, and $(a, d-2)$ are as follows:

$$P[X_{n+1} = (a, d-2) | X_n = (a, d)] = P[Y^{(1)} > Z^{(1)}] \cdot P[Y^{(2)} > Z^{(2)}]$$

$$P[X_{n+1} = (a-2, d) | X_n = (a, d)] = P[Y^{(1)} \leq Z^{(1)}] \cdot P[Y^{(2)} \leq Z^{(2)}]$$

$$P[X_{n+1} = (a-1, d-1) | X_n = (a, d)] = 1 - P[X_{n+1} = (a, d-2) | X_n = (a, d)] - P[X_{n+1} = (a-2, d) | X_n = (a, d)]$$

Since $Y^{(1)}$ and $Z^{(1)}$, $Y^{(2)}$ and $Z^{(2)}$ are independent of each other, by using the probabilities given in Table 2 we obtain:

$$P[Y^{(1)} > Z^{(1)}] = \sum_{y=2}^{6} \sum_{z=1}^{y-1} P[Y^{(1)} = y, Z^{(1)} = z]$$

$$= \sum_{y=2}^{6} \sum_{z=1}^{y-1} P[Y^{(1)} = y] \cdot P[Z^{(1)} = z] = 0.471$$

$$P[Y^{(2)} > Z^{(2)}] = \sum_{y=2}^{6} \sum_{z=1}^{y-1} P[Y^{(2)} = y, Z^{(2)} = z]$$

$$= \sum_{y=2}^{6} \sum_{z=1}^{y-1} P[Y^{(2)} = y] \cdot P[Z^{(2)} = z] = 0.551$$

Therefore,

$$P[Y^{(1)} \leq Z^{(1)}] = 1 - P[Y^{(1)} > Z^{(1)}] = 0.529$$

$$P[Y^{(2)} \leq Z^{(2)}] = 1 - P[Y^{(2)} > Z^{(2)}] = 0.449$$

Thus the state transition probabilities for the case $a \geq 3$ and $d \geq 2$ are as follows:

$$P[X_{n+1} = (a, d-2) | X_n = (a, d)] = P[Y^{(1)} > Z^{(1)}] \cdot P[Y^{(2)} > Z^{(2)}] = 0.259$$

$$P[X_{n+1} = (a-2, d) | X_n = (a, d)] = P[Y^{(1)} \leq Z^{(1)}] \cdot P[Y^{(2)} \leq Z^{(2)}] = 0.237$$

$$P[X_{n+1} = (a-1, d-1) | X_n = (a, d)] = 1 - P[X_{n+1} = (0, d) | X_n = (a, d)] - P[X_{n+1} = (a, d-2) | X_n = (a, d)] = 0.504$$

The state transition probabilities for all other cases can be obtained in the same way; the results are given in Table 3.

FIGURE 1 shows the state transition diagram in the specific situation where the attacker has 6 and the defender has 4 armies. Each case from I to VI given in Tables 1 and 3 is depicted with a rectangle. The state transition probabilities in each case are shown only for a representative state.

TABLE 3 The state transition probabilities

Case	a	d	From state	To state	Transition Probability
I	1	1	$(1,1)$	$(1,0)$ $(0,1)$	0.417 0.583
II	2	1	$(2,1)$	$(2,0)$ $(1,1)$	0.578 0.422
III	≥ 3	1	$(a,1)$	$(a,0)$ $(a-1,1)$	0.659 0.341
IV	1	≥ 2	$(1,d)$	$(0,d)$ $(1,d-1)$	0.254 0.746
V	2	≥ 2	$(2,d)$	$(2,d-2)$ $(0,d)$ $(1,d-1)$	0.152 0.373 0.475
VI	≥ 3	≥ 2	(a,d)	$(a,d-2)$ $(a-2,d)$ $(a-1,d-1)$	0.259 0.237 0.504

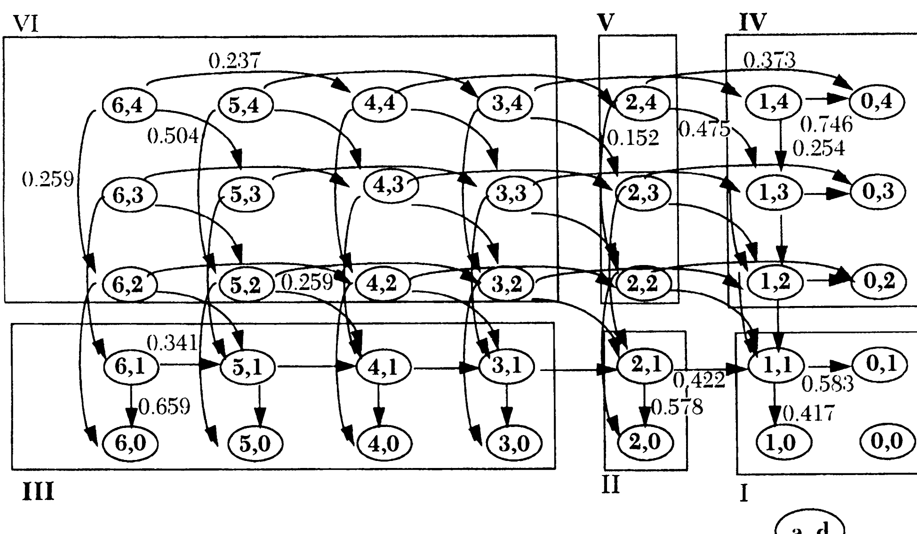

FIGURE 1
The state-transition diagram when the attacker has 6 and the defender has 4 armies.

5. Numerical Results

A computer program was written to generate the state-transition probability matrix and to determine the winning probabilities and expected losses for a given number of initial armies of defender and attacker. In this section, we report the results for each initial state (a, d), $0 < a \leq 30$, $0 < d \leq 30$.

The increase in the dimensions of the state transition matrix is $O(a^2d^2)$, and the increase in the number of elements is $O(a^4d^4)$. For example, if the attacker and defender each has 30 armies, the state space has $31 \times 31 - 1 = 960$ states. Thus the state transition matrix has dimension 960 × 960, and the number of elements in $\underline{P}$ is 921,600,000. This rapid increase in dimension may create computational difficulties in calculating equation (1). However, since the state transition matrix has only a few non-zero elements, numerical techniques for sparse matrices can be exploited to increase computational efficiency and stability.

FIGURE 2 depicts the probabilities that attacker wins with 5, 10, 15, 20, 25, and 30 armies as a function of the number of defender armies. FIGURE 2 shows that when both attacker and defender have the same number of armies, the probability that the attacker wins is below 50%. (This is because in the case of a draw, the defender wins.) When there are twice as many attackers as defenders, the winning probability exceeds 80%.

FIGURE 3 shows expected losses of an attacker with 5, 10, 15, 20, 25, and 30 armies, as a function of the number of defending armies. If the number of attacking armies is twice as many as the defending armies, then the expected loss of an attacker is slightly less than the number of defending armies. For example, if an attacker has 20 and a defender has 10 armies, then attacker wins the war with a probability of 98%, with

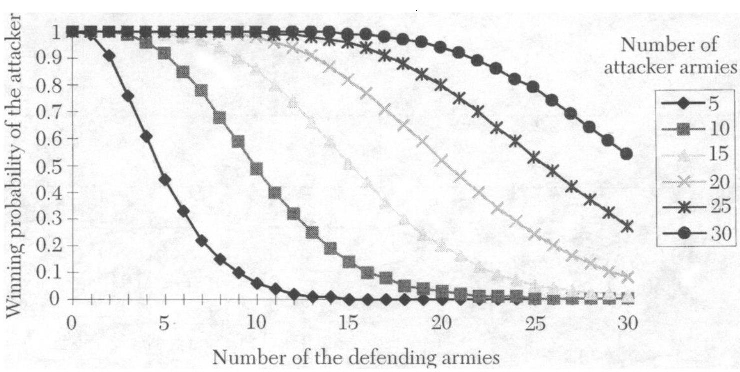

FIGURE 2
The winning probabilities of the attacker.

FIGURE 3 shows expected losses of an attacker with 5, 10, 15, 20, 25, and 30 armies, as a function of the number of defending armies. If the number of attacking armies is twice as many as the defending armies, then the expected loss of an attacker is slightly less than the number of defending armies. For example, if an attacker has 20 and a defender has 10 armies, then attacker wins the war with a probability of 98%, with expected losses of about 9 armies.

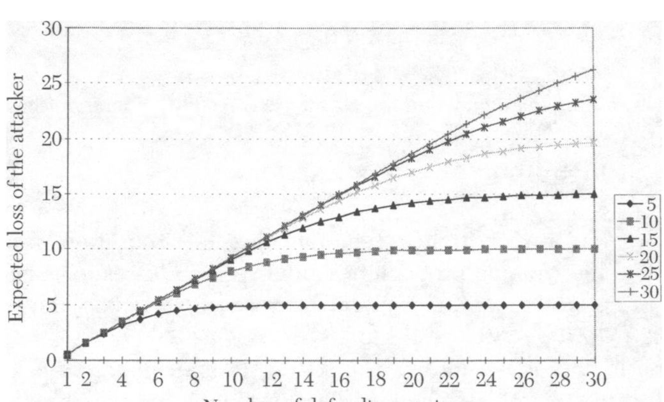

FIGURE 3
Expected loss of the attacker.

In devising a strategy for the game, one considers how many armies will be left in a territory that has just been occupied, since another opponent may attack that territory in the same turn. A simple rule of thumb can be stated as follows: Based on how many defending armies you want to leave on a territory that you want to conquer, attack if you have twice as many armies on a neighbor territory and also if the number of the armies your opponent has is at most half of the number of your armies.

6. Conclusions

One should note that the results above will be useful only over a sequence of many wars. However, since at each turn a player can attack any number of territories and there are many turns in a game, the results may help in devising a useful strategy.

As mentioned in the rules of the game, an attacker may withdraw at any turn. In particular, an attacker who sustains big losses in the course of a war may consider doing so. Determining when to withdraw is another interesting problem; expected loss and winning probability may be used together to devise a strategy. In doing so, one may include the risk characteristics of an individual, leading to different strategies for risk-averse and risk-neutral players. This can be accomplished by using utility functions.

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NOTES

Recurrent Ideas in Number Theory: The Multiple Birkhoff Recurrence Theorem Used to Prove van der Waerden's Theorem

IRENE MULVEY
Fairfield University
Fairfield, CT 06430

In 1977 and 1978, some striking and beautiful connections between the fields of topological dynamics and combinatorial number theory began to emerge. Since then there has been an explosion of sensational results that connect these two fields and provide new and clear insights into some classical number-theoretic theorems and combinatorial questions. This story is about classical number-theoretic results being understood in a new and systematic way using the tools and machinery of dynamical systems. The point of this note is to discuss in some detail one such important number-theoretic result and its new dynamical interpretation, namely van der Waerden's theorem, which provides a particularly clear and simple example. (We focus on the dynamical interpretation of this number-theoretic result. Classical proofs of theorems are not included, but the reader is invited to explore the references **[6]**, **[7]**, **[2]**, **[8]**.)

Number theory is one of the oldest branches of mathematics. It deals with the natural numbers $\mathbb{N}$, which are used for counting and calculations, and it is the solid foundation of modern algebra. Combinatorial number theory, simply speaking, involves questions about sets of natural numbers and classes of sets of natural numbers and the structure of these classes as sets. Questions in combinatorial number theory are quantitative and set-theoretic in nature, as opposed to those concerning algebraic properties of numbers (divisibility, primeness, etc.). In the late 1920's the following problem was posed: If the natural numbers are divided into 2 classes, must there be an arithmetic progression of arbitrary length in one of the classes? Many mathematicians tried to solve this problem—it is appealing due to its (deceptively) simple nature—but the solution by van der Waerden proved to be surprisingly deep. In any event, in 1927 van der Waerden proved the following theorem, answering a slightly more general question.

THEOREM (B. L. van der Waerden, 1927). *For any finite partition of the natural numbers into classes,* $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$, *there is a* C_j *containing an arithmetic progression of arbitrary finite length.*

The original problem and van der Waerden's proof were entirely combinatorial, but we leave combinatorial number theory for a moment and turn to dynamics.

Theoretical dynamical systems was set in motion by Isaac Newton with the development of Calculus and the formulation of the laws of motion. For two centuries, this subject was pursued as a chapter in the theory of differential equations, until a decisive shift in point of view was brought about by Poincaré with his extensive and profound work on celestial mechanics in the late 1800's. This more global analysis is concerned primarily with the geometry of the phase space. In modern dynamical systems, one studies a system that changes over time. Consider as a model a physical system of moving particles or planets governed by some laws (differential equations). A dynamicist observes this system over time and asks questions like the following: Does the system decay? Does the system stabilize? Do the points (particles or planets) return to their original positions or near to their original positions? Such a system has a lot of structure—geometric structure, measure-theoretic structure, etc.—and the point of topological dynamics is to extract and study just the topological structure. There is a rich and beautiful theory to develop which depends only on the topological structure of the system.

More specifically, we will define a dynamical system to be a metric space M and a continuous map $T: M \to M$. We denote this by (M, T). Let $T^0(x) = x$ and for $n \geq 1$, let $T^n(x) = T(T^{n-1}(x))$. For $x \in M$, we define the *orbit* of x to be

$$\mathscr{O}(x) = \{T^n(x) \mid n = 0, 1, 2, \dots\}.$$

If there exists $k \in \mathbb{N}$ such that $T^k(x) = x$, then we say x is *periodic of least period* m where m is the least positive integer such that $T^m(x) = x$. In this case, $\mathscr{O}(x)$ contains exactly m points. A point x is called *eventually periodic* if there exists $k \in \mathbb{N}$ and $T^k(x)$ is periodic. In this case, too, $\mathscr{O}(x)$ is finite. A point $x \in M$ is called *recurrent* for T if and only if there exist a sequence of natural numbers $n_k \to \infty$ and $T^{n_k}(x) \to x$. That is, x returns arbitrarily close to its original position infinitely often.

The idea of recurrence is central in dynamics. Specifically, when the space M of the dynamical system is appropriately bounded, some orbits will necessarily exhibit some form of recurrence, in that they return close to their original position. The first result of this type (for a measure-preserving transformation on a finite measure space) was formulated by Poincaré and is known as the Poincaré Recurrence Theorem. The second result of this kind, more relevant to our purposes, is due to G. D. Birkhoff [**1**].

THEOREM (Birkhoff Recurrence Theorem, 1927). *If X is a compact topological space and T is a continuous map from X to itself, then X has a recurrent point.*

In 1978, H. Furstenberg succeeded in generalizing these two basic recurrence theorems from the framework of a single transformation to that of several commuting transformations. This is a key insight. Let $T_1, T_2, \dots, T_m$ be maps of a metric space M to itself. We say that the maps commute if $T_i \circ T_j = T_j \circ T_i$, for all integers i and j with $1 \leq i, j \leq m$. A point x in M is called *multiply recurrent* (for $T_1, T_2, \dots, T_m$) if there exists a sequence of natural numbers $n_k \to \infty$ with $T_1^{n_k}(x) \to x$, $T_2^{n_k}(x) \to x, \dots$ $T_m^{n_k}(x) \to x$, as $k \to \infty$. The following theorem, due to Furstenberg [**3**], generalizes the Birkhoff Recurrence Theorem since it guarantees the existence of a point that is simultaneously recurrent for all of the commuting transformations.

THEOREM (Multiple Birkhoff Recurrence Theorem, 1978). *If M is a compact metric space and T_1, $T_2, \dots, T_m$ are continuous maps of M to itself which commute, then M has a multiply recurrent point.*

Certainly, the Birkhoff recurrence theorem guarantees for each of the m dynamical systems (M, T_i) that there is a recurrent point. The multiple recurrence theorem says that you can find one point in M that is recurrent simultaneously for every system. That is, there is at least one point in M such that under the action of each of the m transformations, this point returns arbitrarily close to its original position infinitely often, and moreover, for each of the m transformations, the same sequence of return times $\{n_k\}_{k=1}^{\infty}$ will exhibit this recurrence.

The central idea in this note is to show how van der Waerden's theorem follows from the Multiple Birkhoff Recurrence Theorem. First, we prove a preliminary result.

PROPOSITION 1. *Let M be a compact metric space, let $T: M \to M$ be a continuous map, and let $x_0 \in M$. Then, for every $l \geq 1$, and every $\epsilon > 0$, there exist $y \in \mathcal{O}(x_0)$ and $n \geq 1$ which have the property that the points $y, T^n(y), T^{2n}(y), \ldots T^{ln}(y)$ are within ϵ of each other.*

Proof. Let $Y = cl\mathcal{O}(x_0)$, the closure of the orbit of x_0, and let $T_1 = T$, $T_2 = T^2$, $T_3 = T^3, \ldots, T_l = T^l$. Then Y is a compact metric space and the $T_1, T_2, \cdots, T_l$ are a set of commuting transformations from Y to itself. By the Multiple Birkhoff Recurrence Theorem, there exists a multiply recurrent point $y' \in Y$. That is, there exist $n_k \to \infty$ such that $T_1^{n_k}(y') \to y', T_2^{n_k}(y') \to y', \cdots, T_l^{n_k}(y') \to y'$. In particular, for some $n = n_k$, we have $y', T_1^n(y'), T_2^n(y'), \ldots, T_l^n(y')$ all contained in the ball about y' of radius $\epsilon/2$. (We will use the notation $B(y', \epsilon/2)$ for this open ball.) For each $1 \leq i \leq l$, there exists $\gamma_i > 0$ such that $B(T_i^n(y'), \gamma_i) \subseteq B(y', \epsilon/2)$. Now each T_i^n is continuous, so for every γ_i there exists $\delta_i > 0$ such that if $z \in B(y', \delta_i)$, then $T_i^n(z) \in B(T_i^n(y'), \gamma_i)$. If we take $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_1, \epsilon/2\}$, we have if $z \in B(y', \delta)$, then $T_i^n(z) \in B(y', \epsilon/2)$. Now, $y' \in cl\mathcal{O}(x_0)$, so there exists an element $y = T^m(x_0) \in \mathcal{O}(x_0)$ such that $y \in B(y', \delta) \subseteq B(y', \epsilon/2)$. Then for $1 \leq i \leq l$, $T_i^n(y) \in B(y', \epsilon/2)$. This gives us $y = T^m(x_0)$ satisfying $y, T_1^n(y), T_2^n(y), \ldots, T_l^n(y)$ are all within ϵ of each other. But, $T_i^n(y) = T^{in}(y)$, so $y, T^n(y), T^{2n}(y), \ldots, T^{ln}(y)$ are all within ϵ of each other, as needed. ■

Next we define a specific dynamical system to which we will apply Proposition 1. (The following is a standard construction in dynamical systems.) Take the finite set of symbols $\Gamma = \{1, 2, \ldots, q\}$ and form the space $X = \Gamma^{\mathbb{N}}$ of all infinite, one-sided sequences with entries from Γ. So, $x \in X$ is of the form $x = (x_0, x_1, x_2, \ldots)$ where $x_i \in \Gamma$. Define a metric d on X by $d(x, x) = 0$ and if $x \neq y$, then $d(x, y) = 1/(k+1)$, where k is the smallest integer such that $x_k \neq y_k$. The set X with metric d is a compact metric space. (To see this, one can prove that every sequence in the metric space X has a convergent subsequence, using a standard diagonalization argument. Alternatively, one can observe that X is the Cartesian product of Γ with itself countably many times, and that each Γ, viewed as a finite set with the discrete topology, is compact. Note that the product and metric topologies agree.) Now we apply Proposition 1 to the dynamical system (X, T). Take any $x \in X$ and take $\epsilon < 1$. Then, for any $l \geq 1$, there exist $y \in \mathcal{O}(x)$ and $n \geq 1$ such that $y, T^n(y), T^{2n}(y), \ldots, T^{ln}(y)$ are all infinite one-sided sequences that agree in the first entry. Thus we have shown:

PROPOSITION 2. *For any $x \in X$ and any $l \geq 1$, there exists $m \geq 1$ such that $x_m = x_{m+n} = x_{m+2n} = \ldots = x_{m+ln}$.*

Proof of van der Waerden's theorem. Take $l \in \mathbb{N}$ and let $\mathcal{P}$ be a partition of $\mathbb{N}$ into q sets: $B_1, B_2, \ldots, B_q$. Now $\mathcal{P}$ determines an element $x = (x_1, x_2, x_3, \ldots) \in \Gamma^{\mathbb{N}}$ as follows: $x_j = i$ where $j \in B_i$. By Proposition 2, there exists $m \geq 1$ such that $x_m = x_{m+n} = x_{m+2n} = \ldots x_{m+ln}$. That is, the integers $m, m+n, m+2n, \ldots, m+ln$ are all contained in the same B_i. Thus B_i contains an arithmetic progression of length $l+1$, and the proof is complete. ■

In the preceding proof, a classical number-theoretic result is seen in a completely new light by applying techniques of topological dynamics. In fact, this is only the beginning. The tools and techniques of dynamical systems can be applied to many results of this type. Many number-theoretic results have the following form: For any finite partition of the natural numbers into classes, at least one class has property P. If P is the property that a subset contains an arithmetic progression of arbitrary finite length, we have the theorem of van der Waerden. Here is another number-theoretic result that can be viewed from a dynamical perspective:

THEOREM (I. Schur, 1916). *For any finite partition of the natural numbers into classes, one can find two numbers u and v in one of the classes* C_j *such that the sum* $(u + v)$ *also belongs to* C_j.

The machinery that has been developed and continues to be developed has proved useful in tackling this and other problems in combinatorial number theory; it also suggests problems of independent interest in dynamical systems. The interested reader is invited to investigate the references, in particular, the beautiful research monograph by H. Furstenberg [3].

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Functions with Compact Preimages of Compact Sets

JENNIFER BRYANT
JAMES KUZMANOVICH
ANDREY PAVLICHENKOV[1]
Wake Forest University
Winston-Salem, NC 27109

This article originated with what seemed to be a straightforward question asked during a Topology class. A standard way of defining continuity is in terms of open and closed sets; a function is continuous if and only if preimages of open sets are open, or equivalently, if and only if preimages of closed sets are closed. What can we say if we assume that a function has the seemingly stronger property that preimages of compact sets are compact? We call such a function *preimage-compact*. An immediate exercise is to determine whether a relationship exists between preimage-compact functions and continuous functions. Consideration of any constant function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = c$ for all x, shows that a continuous function need not be preimage-compact, for $\{c\}$ is compact but $f^{-1}(\{c\}) = \mathbb{R}$ is not. It might seem likely that preimage-compact functions are continuous, but this is not the case, for

$$f(x) = \begin{cases} x + \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{1}$$

is preimage-compact, but not continuous.

In what follows we study preimage-compact functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$. This exploration is intended to find out how strong this property is and to determine how large the set of discontinuities of such functions can be. We show that preimage-compact real-valued functions defined on the real line are continuous on a dense open set. On the other hand, we give examples to show that the set of discontinuities of a preimage compact function can be large in a measure theoretic sense.

We found that what seemed to be a simple question asked during a Topology class led us to an intriguing exploration that made illustrative use of several course results, including the Baire Category Theorem and the Cantor set. Since our exploration combines and extends several important concepts, we hope that instructors find this paper to be appropriate for use in beginning Topology classes or Advanced Calculus classes, either in its entirety for a reading assignment or in outline form as a project.

We first establish some preliminary facts about sets of continuity and discontinuity of preimage-compact functions. The following proposition shows that a preimage-compact function must be unbounded in every neighborhood of a discontinuity. The discontinuity of the function (1) is typical for preimage-compact functions; preimage-compact functions cannot have simple 'jump discontinuities.'

PROPOSITION. *Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be preimage-compact. If $f(x)$ is bounded on an open interval I, then $f(x)$ is continuous on I.*

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[1] The authors were members of a topology class, with James Kuzmanovich as instructor.

Proof. Suppose $f(I) \subseteq K$ for a compact set K. Note $f|_I : I \longrightarrow K$. If C is a closed subset of K, then C is compact, and thus by hypothesis $f^{-1}(C)$ is compact and consequently closed. Since $f|_I^{-1}(C) = f^{-1}(C) \cap I$ is closed in I for the arbitrary closed set C, we have that preimages of closed sets under $f|_I$ are closed; this is equivalent to the continuity of $f|_I$. Since I is open, $f(x)$ is continuous on I. ■

Let $\mathscr{C} = \{c \in \mathbb{R}: f \text{ is continuous at } c\}$. We have the following immediate corollary to the Proposition.

COROLLARY. *If $f(x)$ is preimage-compact, then the set $\mathscr{C}$ of points of continuity of $f(x)$ is open.*

Proof. Let $c \in \mathscr{C}$. Since $f(x)$ is continuous at c, there exists I_c around c such that $f(x)$ is bounded on I_c; hence by the Proposition $\mathscr{C}$ contains an open interval about each of its points and consequently is open. ■

Let $\mathscr{D} = \mathbb{R} \setminus \mathscr{C}$ be the set of discontinuities of $f(x)$. If $f(x)$ is preimage-compact, then how big can $\mathscr{D}$ be? How small can $\mathscr{C}$ be? Must $\mathscr{C}$ be nonempty? A partial answer is given in the next Theorem. First recall that a set S is called *dense* if $\overline{S} = \mathbb{R}$ where $\overline{S}$ denotes the closure of S. On the other hand S is called *nowhere dense* if $\overline{S}$ has empty interior. We will make strong use of the Baire Category Theorem (see **[3]**, page 294, or **[1]**, page 144); it states that no open subset of $\mathbb{R}$ is contained in the union of countably many closed nowhere dense subsets.

THEOREM. *If $f(x)$ is preimage-compact, then $f(x)$ is continuous on a dense open set, and the set of discontinuities of $f(x)$ is nowhere dense.*

Proof. The set $\mathscr{C}$ of points at which $f(x)$ is continuous is open by the Corollary. Let $D_n = f^{-1}([-n, n]) \cap \mathscr{D}$. Since $\mathscr{C}$ is open, $\mathscr{D}$ must be closed, and since $f(x)$ is preimage-compact, $f^{-1}([-n, n])$ must be compact and hence closed; it follows that D_n is closed. Then D_n must be nowhere dense, for if D_n were to contain an open interval I, then $f(x)$ would be bounded on I and hence continuous on I by the Proposition. Thus $\mathscr{D} = \cup D_n$ must be nowhere dense by the Baire Category Theorem. It follows that $\mathscr{C}$ is dense. ■

Remark. Readers with more knowledge of topology should note that the Theorem holds, with essentially the same proof, for preimage-compact functions f: $X \longrightarrow Y$ if X is a locally compact Hausdorff space and Y is a space that is a countable union of compact subspaces. Considering any one-to-one function from $\mathbb{R}$ into an uncountable discrete space will show that the countable union hypothesis is needed.

The Theorem shows that the set $\mathscr{D}$ of discontinuities of $f(x)$ is in some sense small. The following example shows that $\mathscr{D}$ can be uncountable. The exercises that follow ask the reader to extend the example and show that $\mathscr{D}$ can have positive, or even infinite, measure.

Example. For an open interval (a, b) and a positive integer n, we let $f_{a,b,n}(x) = \cot^2\left((x-a)\frac{\pi}{b-a}\right) + n$. Observe that $f_{a,b,n}(x)$ is well-defined on (a, b), has minimum value n, is concave up, and has vertical asymptotes at $x = a$ and $x = b$.

Recall the description of the Cantor set $\mathscr{G}$ given in (**[3]**, page 177). We let $A_0 = [0, 1]$, and let A_1 be the set obtained from A_0 by deleting the open middle third $(\frac{1}{3}, \frac{2}{3})$. Inductively we continue to delete middle thirds by letting

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right).$$

Then $\mathscr{G} = \bigcap_{n=0}^{\infty} A_n$. The Cantor set is a compact, totally disconnected, perfect (and hence uncountable) subset of $[0, 1]$. Denote the union of the deleted middle third intervals by T; of course, $T = [0, 1] \setminus \mathscr{G}$.

We define a function f: $\mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = f_{a,b,n}(x)$ if x is in one of the middle third intervals $(a, b) = \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)$ deleted at the n^{th} stage; otherwise, we let $f(x) = x$. The graphs of the first few $f_{a,b,n}(x)$ portions of $f(x)$ are given in FIGURE 1.

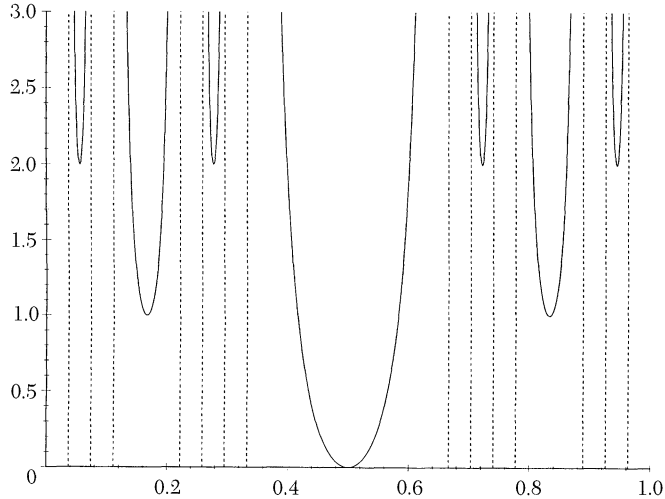

FIGURE 1
Graphs of the first $f_{a,b,n}(x)$.

Observe that $f(x)$ is continuous for every $x \in \mathbb{R} \setminus \mathscr{G}$. On the other hand, if $c \in \mathscr{G}$, then every neighborhood $(c - \epsilon, c + \epsilon)$ of c contains a deleted middle third interval of the form $(a, b) = \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)$ for some n and k; since $f(x) = f_{a,b,n}(x)$ is unbounded on the open interval (a, b), $f(x)$ is not continuous at c. Thus the set $\mathscr{D}$ of discontinuities of $f(x)$ is exactly $\mathscr{G}$. It remains to show that $f(x)$ is preimage-compact. Suppose C is a compact subset of $\mathbb{R}$. Since C is bounded above, $f_{a,b,n}^{-1}(C)$ is empty for all but finitely many $f_{a,b,n}(x)$; if $f_{a,b,n}^{-1}(C)$ is nonempty, then the definition of $f_{a,b,n}(x)$ and the shape of its graph show that $f_{a,b,n}^{-1}(C)$ is compact. Since $f(x) = x$ on $\mathbb{R} \setminus T$, $f|_{\mathbb{R}\setminus T}^{-1}(C) = C \cap (\mathbb{R} \setminus T)$ is compact. Hence $f^{-1}(C) = (C \cap (\mathbb{R} \setminus T)) \cup \left(\bigcup f_{a,b,n}^{-1}(C)\right)$ is a finite union of compact sets, and hence is compact; consequently, $f(x)$ is preimage-compact. ■

We close with two exercises for the reader.

Exercise 1. Show that there exists a preimage-compact function $f(x)$ such that the set $\mathscr{D}$ of discontinuities of $f(x)$ has positive measure. Hint: modify the preceding Example by using a Cantor set with positive measure (for example, see **[2]**, page 88).

Exercise 2. Show that there exists a preimage-compact function $f(x)$ whose set $\mathscr{D}$ of discontinuities has infinite measure.

Acknowledgment. The authors are grateful to the referees for their helpful suggestions.

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Volume and Surface Area for Polyhedra and Polytopes

JOHN EMERT
ROGER NELSON
Ball State University
Muncie, IN 47306-0490

When the curious calculus student first realizes the relationship between the area and circumference of the circle, $dA/dr = C$, and the similar relationship between the volume and surface area of the sphere, $dV/dr = A$, the question arises: "Is this always the case?" We show that, indeed, such a relation exists in all dimensions for (convex) regular polytopes when the derivative is taken with respect to the *inner radius*, i.e., the (minimal) distance from the center to the boundary. With some modification, a similar relation holds for all polytopes. This approach leads to a related, turn-of-the-century result due to Minkowski.

1. Introductory examples The circle and sphere are generalized by the $(n-1)$-dimensional hypersphere, defined as the set of points in n-space that are a constant distance r from the center point. The *content* (n-dimensional volume) bounded by a hypersphere of radius r is known to be (see [**4**], for example)

$$V = \frac{2r^n\pi^{n/2}}{n\Gamma(n/2)},$$

where Γ is the gamma function. It is a standard exercise (see [**3**, p. 125]) to verify that the derivative dV/dr is the content of the bounding hypersphere.

Using standard trigonometry, a regular k-gon with inner radius r can be shown to have circumference $C = 2rn\tan(\pi/n)$ and area $A = r^2 n\tan(\pi/n)$. Note that $dA/dr = C$.

A hypercube with inner radius r has side length $2r$, so the content of an n-dimensional hypercube is $V = 2^n r^n$. The surface of an n-dimensional hypercube has $2n$ faces, each an $(n-1)$-dimensional hypercube. Therefore, an n-dimensional hypercube has surface content $A = 2n \cdot 2^{n-1} r^{n-1} = n2^n r^{n-1}$. Again, as predicted, $dV/dr = A$.

It is well known that there are exactly five convex regular polyhedra in dimension 3, the Platonic solids. In dimension 4, there are exactly six convex regular polytopes. In dimensions $n \geq 5$, there are exactly three regular polytopes: the hypercube, regular simplex, and cross polytope (see, e.g., [**3**, p. 136]). After introducing and exploring the regular n-dimensional simplex using standard content formulae expressed *in terms of edge length*, we give a separate argument that verifies the "volume-surface area" relationship for *all* regular n-polytopes.

2. The regular n-simplex The regular n-dimensional simplex Δ_n, determined by $n+1$ points arranged equidistantly in n-space, enjoys the expected relationship between content and surface content. We have already observed this relationship in dimension 2, for the equilateral triangle. The proof for all dimensions will use the following lemma.

LEMMA. *For a regular n-dimensional simplex* Δ_n $(n \geq 2)$, *the inner radius* r, *altitude* a, *and edge length* e *satisfy*

$$a = (n+1)r \quad \text{and} \quad e = \sqrt{2n(n+1)}\, r.$$

Proof. The result (easily verified when $n = 2$) is shown by induction on the dimension n. Assuming the result for dimension n, construct a triangle in Δ_{n+1} determined by a vertex, the center of the opposite n-dimensional face, and the center of a common $(n-1)$-dimensional face, as shown in FIGURE 1. This right triangle has an altitude of Δ_{n+1} and altitude and radius of n-dimensional faces as edges; label their lengths as a_{n+1}, a_n, and r_n, respectively. The altitudes pass through the center O of Δ_{n+1}; mark this point and drop a perpendicular to the opposite triangle edge, piercing the center of the n-dimensional face as shown in FIGURE 2. The length of the constructed radius of Δ_{n+1} is r_{n+1}, as marked.

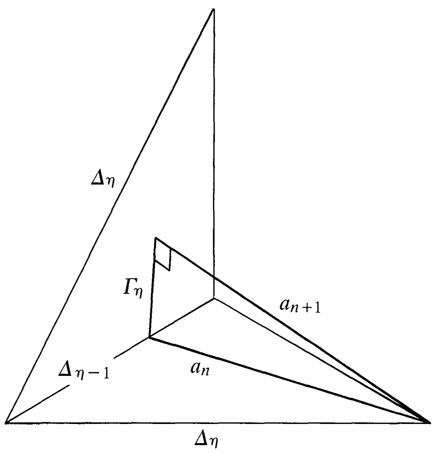

FIGURE 1
The altitudes of Δ_{n+1} and Δ_n.

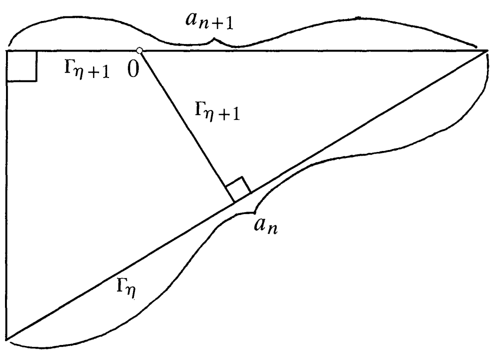

FIGURE 2
The center O and inner radius r_{n+1} of Δ_{n+1}.

By similarity of triangles,

$$\frac{r_{n+1}}{a_n - r_n} = \frac{r_n}{a_{n+1}}.$$

By the induction hypothesis, $r_{n+1}a_{n+1} = r_n(nr_n)$, so

$$r_n^2 = \frac{r_{n+1}a_{n+1}}{n}. \tag{1}$$

Since $a_{n+1}^2 = a_n^2 - r_n^2$, we use the induction hypothesis $a_n = (n+1)r_n$ to conclude that $a_{n+1}^2 = (n+1)^2 r_n^2 - r_n^2$. By (1), it follows that $a_{n+1}^2 = (n+2)r_{n+1}a_{n+1}$. Therefore,

$$a_{n+1} = (n+2)r_{n+1}, \tag{2}$$

and the first part of the lemma is shown.

To verify the second equality, substitute (2) and the induction hypothesis into (1), and solve for r_{n+1}. ■

An n-simplex is the n-dimensional cone over an $(n-1)$-simplex. By induction, the content V of Δ_n with edge length e is therefore

$$V = \frac{1}{n}a_n V' = \frac{\sqrt{n+1}}{n!\sqrt{2^n}}e^n,$$

where V' is the content of Δ_{n-1}. By the chain rule and the preceding lemma, the derivative of V with respect to the inner radius is

$$\begin{aligned}\frac{dV}{dr_n} &= \frac{\sqrt{n+1}}{n!\sqrt{2^n}}ne^{n-1}\frac{de}{dr_n}\\ &= \frac{\sqrt{n+1}}{n!\sqrt{2^n}}ne^{n-1}\sqrt{2n(n+1)}\\ &= (n+1)\frac{\sqrt{n}}{(n-1)!\sqrt{2^{n-1}}}e^{n-1},\end{aligned}$$

which, as the content of $n+1$ simplices of edge length e and dimension $n-1$, is exactly the surface content of Δ_n.

The following formula-free argument yields the same result. One may view Δ_n as being constructed from $n+1$ smaller (non-regular) simplices created by coning from the centroid p of Δ_n to its *principal* ($(n-1)$-dimensional) faces. Dilation of Δ_n from its centroid corresponds to dilation of each smaller simplex from vertex p to its opposite face; the inner radius r_n of Δ_n is the height of each smaller simplex. Hence, the derivative of the content of a cone over a principal face with respect to its height is the content of that principal face. This relationship—which is as natural as the volume-surface relationship for the sphere which motivated this inquiry—holds for cones over any principal face in any dimension. It is the key to the argument for all regular polyhedra and polytopes.

3. The regular case Suppose Γ_n is a cone with height h over a principal face Γ_{n-1}. If Γ_n is dilated from the cone point, then the content of Γ_{n-1} is a function of h given by γh^{n-1} for some constant γ. The content of Γ_n is given, however, by the familiar formula $\frac{1}{n}hF_{n-1}$, where F_{n-1} is the content of Γ_{n-1}. Like its special cases for the area of a triangle and the volume of a 3-dimensional cone, this formula may be found by integrating the cross sectional content γh^{n-1} of Γ_n along its altitude. Hence, the content of Γ_n is $\frac{1}{n}\gamma h^n$. The next lemma follows immediately:

LEMMA. *Let Γ_n be a cone of height h over principal face Γ_{n-1}. The derivative of the content of cone Γ_n with respect to its height h is the content of the principal face Γ_{n-1}.*

THEOREM 1. *The derivative of the content of any convex regular n-dimensional polytope with respect to its inner radius is the content of the boundary of the polytope.*

Proof. Any convex regular polytope P is the union of κ identical cones Γ_n with principal faces Γ_{n-1} opposite the cone point and height r, which is the inner radius of the regular polytope, as illustrated in FIGURE 3. The content of P is $V = \kappa\frac{1}{n}\gamma r^n$, so that $dV/dr = \kappa\gamma r^{n-1}$, the content of the boundary of P. ■

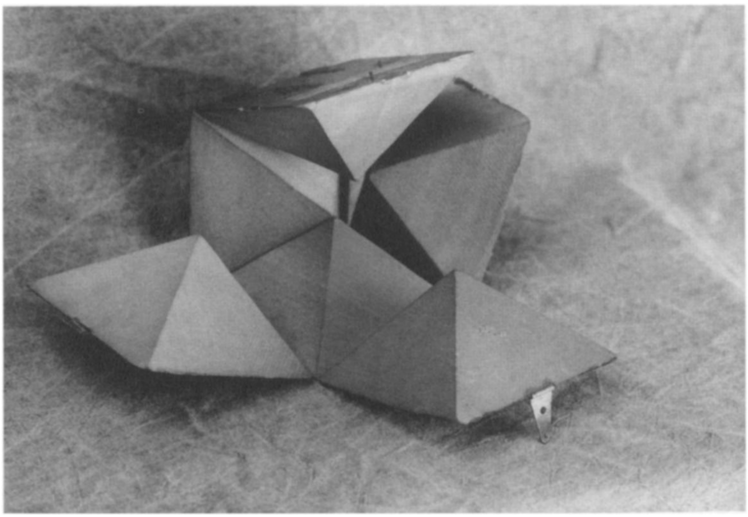

FIGURE 3
The cube as a union of six simplices.

For some convex sets, $dV/dr = \xi A$, where ξ is a constant not equal to 1. Consider, for example, the ellipse formed by stretching a circle in one direction by a factor of $a > 1$. Area increases by the factor a while arc length increases by a factor b ($b \approx \sqrt{\frac{1+a^2}{2}}$ by approximating an elliptic integral) which is strictly less than a. Hence the derivative of area with respect to inner radius (length of the semi-minor axis) is $\frac{a}{b}$ $\left(\frac{a}{b} \approx \sqrt{\frac{2a^2}{1+a^2}}\right)$ times the perimeter. One can also show that the derivative of the area of a rectangle with length l and width w $(l \geq w)$ with respect to the distance from the center to the near side is $\frac{2l}{w+l}$ times the perimeter. When the ellipse is a circle or the rectangle is a square, the constant $\xi = 1$. In general, when is ξ equal to 1?

4. The circumscribing polytope case Let V be the content of a polytope P having faces with content A_i. We cone from an arbitrary point p of the polytope P and let a provisional inner radius r be the distance to a nearest face. This inner radius coincides with the height of the cone over this face, which has content A_0. Then there are constants a_i and k_i such that $A_i = a_i A_0$ and $h_i = k_i r$, where h_i is the height of the ith cone. These relations hold, of course, for any *dilation* of the polytope P. Since h_i is the distance from p to the hyperplane determined by a principal face, $k_i \geq 1$. Note that the total surface content A of the polytope is then $(\Sigma a_i)A_0$. Furthermore, if V_i is the content of the ith cone, then

$$\frac{dV}{dr} = \sum \frac{dV_i}{dr} = \sum \frac{dV_i}{dh_i} = \sum k_i A_i$$
$$= \sum k_i a_i A_0 = \frac{\Sigma a_i k_i}{\Sigma a_i} \left(\sum a_i \right) A_0$$
$$= \frac{\Sigma k_i a_i}{\Sigma a_i} A.$$

Some choice of coning point minimizes the constant $\xi = \frac{\Sigma k_i a_i}{\Sigma a_i}$ at a value greater than or equal to 1. We have $\xi = 1$ if and only if $k_i = 1$ for each i, which is equivalent to the statement that P circumscribes an $(n-1)$-sphere centered at the coning point p. We have shown the following result.

THEOREM 2. *Suppose P is a polytope with content V and surface content A. Suppose also that P undergoes a dilation centered at an interior point p. Let r be the distance from p to the boundary of P. Then $dV/dr = A$ if and only if the sphere centered at p with radius r is circumscribed by P.*

The polytopes that can circumscribe a sphere include the regular polytopes, the semiregular polyhedra and their duals, and many other non-regular polytopes, such as in FIGURE 4. In general it is not possible to pick an "inner radius" r so that $dV/dr = A$. But is there a different variable that can replace r to achieve this "volume-area" relationship?

5. The general case Adopting a new point of view toward the circumscribing polytope case enables us to generalize the "volume-surface area" relationship to all polytopes. When each principal face is the same distance r from a central point p, multiplying r by a fixed proportion has the same effect as adding a fixed amount ϵ to r. Theorem 2 tells us that this coincidence occurs only for circumscribing polytopes. We adopt this latter point of view for the general situation.

Given a polytope P, choose any point p in the interior of P and a positive ϵ. Suppose P expands by pushing out each principal face Γ a distance ϵ to a parallel face $\tilde{\Gamma}$. See FIGURE 5. In general this expansion is not a dilation since the resulting polytope may not be similar to the original. (By this process each non-principal face will typically be carried to different locations by each of the principal faces containing it. As these faces will not align, as in FIGURE 6, the resulting shears must be patched.) Denote the resulting "ϵ-collared polytope" of P as $P_{\epsilon, p}$.

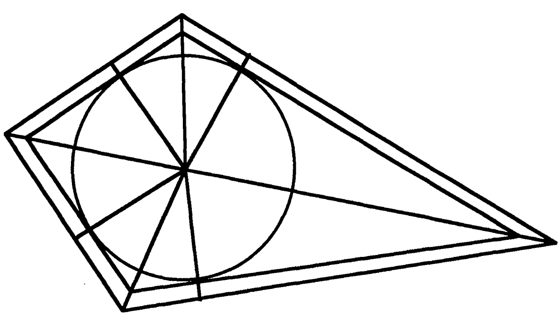

FIGURE 4
A non-regular polytope can circumscribe a sphere.

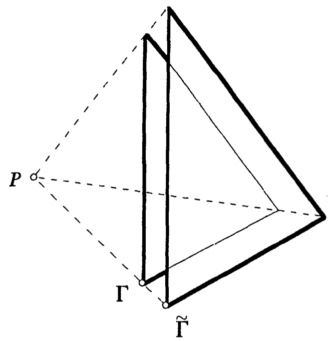

FIGURE 5
Pushing out each principal face.

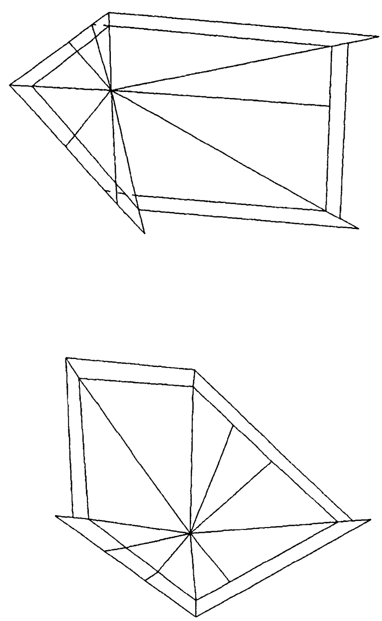

FIGURE 6
Pushing can create sheared principal faces.

THEOREM 3. *Suppose P is a polytope with content V and surface content A. Let p be any point in P. Suppose P expands by pushing each principal face out ϵ from p to form an ϵ-collared polytope $P_{\epsilon,p}$. Then $dV/d\epsilon = A$.*

Proof. Let h_i be the distance from p to the i-th principal face, as before, and note that $\frac{d\epsilon}{dh_i} = 1$. Then

$$\frac{dV}{d\epsilon} = \sum \frac{dV_i}{d\epsilon} = \sum \frac{dV_i}{dh_i} = \sum A_i = A. \qquad \blacksquare$$

The (non-polytope) ϵ-neighborhood of P, $P_\epsilon = \{x \mid d(x, P) \leq \epsilon\}$, can also be formed. Compare the example in FIGURE 7 to the polytope collarings of FIGURE 6. For a fixed polytope P and constant ϵ, P_ϵ and the various $P_{\epsilon,p}$ are all distinct, but the differences in their contents are of the order ϵ^n.

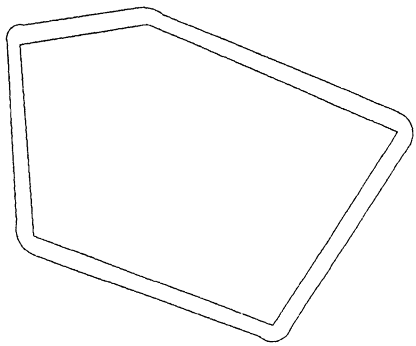

FIGURE 7
An ϵ-neighborhood of a polytope.

If $\mathscr{V}(P)$ represents the content of P, Theorem 3 implies that

$$\lim_{e \to 0} \frac{\mathscr{V}(P_{\epsilon,p}) - \mathscr{V}(P)}{\epsilon} = A.$$

In a 1901 article in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Minkowski used ϵ-neighborhoods to show that

$$\lim_{e \to 0^+} \frac{\mathscr{V}(P_{\epsilon}) - \mathscr{V}(P)}{\epsilon} = A.$$

See [**1**, II, p. 550] or [**2**, II, p. 18] for Minkowski's alternate approach to this result.

With modifications, Theorem 3 can be extended to other regions, such as *starlike* polygons and their n-dimensional analogues. Recall that a region R is starlike if there is some point p in the interior of R (a *star point*) that can see, through R, every point of R. Define $R_{\epsilon,p}$, where p is a star point, to be an ϵ-collaring of R, as before. In this setting the argument for Theorem 3 extends to starlike polygons and their n-dimensional analogues. Minkowski's approach can also extend to such regions if certain overlapped regions are considered "signed volume."

The results of Theorem 3 and Minkowski can also be extended to general (curved) convex sets. A convex set C can be approximated arbitrarily closely by polytopes $\{P_i\}$ so that the content and surface content of the P_i tend in the limit to the content and surface content of C. Similarly, the ϵ-neighborhood C_ϵ is approximated by ϵ-collarings or ϵ-neighborhoods of the P_i. The polytope approximations show that Theorem 3 extends to convex sets.

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On the Fermat–Torricelli Points of Tetrahedra and of Higher Dimensional Simplexes

SADI ABU-SAYMEH
MOWAFFAQ HAJJA
Yarmouk University
Irbid, Jordan

Introduction It is well known (see, e.g., [**6**, pp. 24–34], [**2**, p. 22], [**5**]) that if ABC is a non-collinear triangle, then there exists a unique point P in the triangle that minimizes the quantity $XA + XB + XC$. If any of the vertices of ABC holds an angle greater than or equal to $2\pi/3 = 120°$, then P coincides with that vertex. Otherwise, P lies inside the triangle and satisfies the equiangular property

$$\angle APB = \angle BPC = \angle CPA = 2\pi/3.$$

The point P is called the *Fermat point* (or perhaps more duly [**4**], the *Fermat–Torricelli point*) of ABC.

Similar results hold for tetrahedra. A non-planar tetrahedron $ABCD$ again has a unique (Fermat) point P that minimizes the quantity $XA + XB + XC + XD$. If any of the vertices holds a solid angle (whose "measure" is) at least π, then P coincides with that vertex. Otherwise, P lies inside the tetrahedron and satisfies the equiangular property

$$|\langle P; BCD\rangle| = |\langle P; ACD\rangle| = |\langle P; ABD\rangle| = |\langle P; ABC\rangle|.$$

Here $\langle P; BCD\rangle$ stands for the solid angle enclosed in the rays PB, PC, PD. Its measure $|\langle P; BCD\rangle|$ is given by the area of the spherical triangle whose vertices are the points where PB, PC, PD intersect the unit sphere centered at P ([**3**], [**4**]).

These results on tetrahedra were discovered more than a century ago by Lindelöf [**7**, pp. 191–203]. Although they were recently polished and included in the recent Swedish article [**4**], the results do not seem to be well enough known. This claim is illustrated by various false or misleading statements in the literature; see, e.g., the naive questions given (by one of the present authors!) at the end of [**1**]. Also, Lindelöf's proofs rely heavily on facts from spherical trigonometry and include a dose of visual geometric thinking heavy enough to discourage even the interested from generalizations to higher dimensions.

In this note, we give a complete but elementary treatment of this problem for a general n-hedron. Unaware of Lindelöf's results, we found that the appropriate measure to use for the solid angle $\langle P; A_1\ A_2\ A_3\rangle$ is given by the average

$$\frac{\cos\angle A_1PA_2 + \cos\angle A_2PA_3 + \cos\angle A_3PA_1}{3}$$

of the cosines of its ordinary subangles. Calling this quantity $\cos\langle P; A_1A_2A_3\rangle$, we prove that if any vertex of a tetrahedron holds an angle whose cosine is less than or equal to $-1/3$, then the Fermat point P coincides with that vertex. Otherwise, P is interior and satisfies the equiangular property

$$\cos\langle P; BCD\rangle = \cos\langle P; ACD\rangle = \cos\langle P; ABD\rangle = \cos\langle P; ABC\rangle.$$

This formulation has the advantage of having a transparent analogue for higher-dimensional simplexes. Moreover, the method used in obtaining this result for a tetrahedron is purely algebraic and works equally well in all dimensions.

Definitions Let m and n be any natural numbers with $n \geq 3$. An *n-hedron* (in $\mathbb{R}^m$) is defined to be any set $H = \{A_1, \dots, A_n\}$ of n points in $\mathbb{R}^m$. (Thus 3-hedra are triangles, 4-hedra are tetrahedra, etc.) The n-hedron $H = \{A_1, \dots, A_n\}$ is called *non-collinear* if its vertices $A_1, \dots, A_n$ are non-collinear. (We have not used the more familiar term "simplex" because we do not want to impose any conditions of linear independence on the edges.)

An *n-hedral angle* (in $\mathbb{R}^m$) is defined to consist of a vertex $B \in \mathbb{R}^m$ and a set $L = \{B_1, \dots, B_n\}$ of points in $\mathbb{R}^m$ with $B_i \neq B$ for $i = 1, \dots, n$. Such an angle will be denoted by

$$\langle B; L \rangle = \langle B; \{B_1, \dots, B_n\} \rangle.$$

If L' is a subset of L, then the angle $\langle B; L' \rangle$ is called a *subangle* of $\langle B; L \rangle$. In particular, a 2-hedral angle $\beta = \langle B; \{B_1, B_2\} \rangle$ is thought of as the ordinary angle $\angle B_1 B B_2$ and its cosine is defined in the usual way:

$$\cos\langle B; \{B_1, B_2\} \rangle = \frac{(B_1 - B) \cdot (B_2 - B)}{\| B_1 - B \| \| B_2 - B \|},$$

where $\|(x_1, \dots, x_m)\| = \sqrt{x_1^2 + \cdots + x_m^2}$. The *cosine* of an n-hedral angle $\beta = \langle B; \{B_1, \dots, B_n\} \rangle$ is then defined as the average of the cosines of its 2-hedral subangles:

$$\cos\beta = \frac{\sum_{1 \leq i < j \leq n} \cos\angle B_i B B_j}{\binom{n}{2}} = 2\frac{\sum_{1 \leq i < j \leq n} \cos\angle B_i B B_j}{n(n-1)}. \tag{1}$$

(Simple calculations show that $\cos\beta$ is the average of the cosines of its r-hedral subangles for all r with $2 \leq r \leq n$. This fact is not used in the sequel.)

A simple lemma The following easy lemma will be very useful in proving our main results. It locates the point X within a given n-hedral angle $\langle B; \{B_1, \dots, B_n\} \rangle$ that maximizes the quantity $\sum_{i=1}^n \cos\angle B_i B X$.

LEMMA 1. *Let $\beta = \langle B; \{B_1, \dots, B_n\} \rangle$ be an n-hedral angle. For $X \neq B$, let*

$$g(X) = \sum_{i=1}^{n} \cos\angle B_i B X.$$

Then $g(X) \leq [n + n(n-1)\cos\beta)]^{1/2}$, and equality holds if and only if for some $t > 0$,

$$X - B = t \sum_{i=1}^{n} \frac{B_i - B}{\| B_i - B \|}.$$

Proof. Let $U_1, \dots, U_n$ and V be defined by

$$U_i = \frac{B_i - B}{\| B_i - B \|} \quad \text{and} \quad V = \frac{X - B}{\| X - B \|},$$

and let

$$G(V) := g(X) = \sum_{i=1}^{n} V \cdot U_i = V \cdot \sum_{i=1}^{n} U_i.$$

It follows from the Cauchy–Schwartz inequality that $V_0 = (\Sigma_{i=1}^{n} U_i)/\|\Sigma_{i=1}^{n} U_i\|$ is the unique unit vector V that maximizes $G(V)$, and that

$$G(V_0) = \left\| \sum_{i=1}^{n} U_i \right\| = \left[\left(\sum_{i=1}^{n} U_i \right) \cdot \left(\sum_{i=1}^{n} U_i \right) \right]^{1/2}$$
$$= \left[n + 2 \sum_{1 \le i < j \le n} U_i \cdot U_j \right]^{1/2} = \left[n + 2 \sum_{1 \le i < j \le n} \cos \angle B_i B B_j \right]^{1/2}$$
$$= \left[n + n(n-1)\cos\beta \right]^{1/2}.$$

(The last equality follows from Equation (1).) This proves our lemma.

Since $G(V_0) \ge 0$, the last line of the proof implies the following inequality.

COROLLARY 2. *If β is any n-hedral angle, then* $\cos\beta \ge -1/(n-1)$.

The main results In what follows, we fix a non-collinear n-hedron $H = \{A_1, \dots, A_n\}$ in $\mathbb{R}^m$. For $1 \le i \le n$, we define the *A_i-face* A_i^* of H to be the $(n-1)$-hedron obtained by deleting A_i from $\{A_1, \dots, A_n\}$ and we define the *vertex angle* α_i by $\alpha_i = \langle A_i; A_i^* \rangle$. We are interested in minimizing the function

$$f(X) = \sum_{i=1}^{n} \|X - A_i\|.$$

Since $f(X) \to \infty$ as $\|X\| \to \infty$, we may restrict the domain of f to a compact set. It follows that f attains its minimum value. Also, if P and Q are distinct points and if M is the midpoint of the line segment PQ, let N be the point on the extension of the line segment A_iM so that $\|A_i - M\| = \|N - M\|$. (See FIGURE 1.)

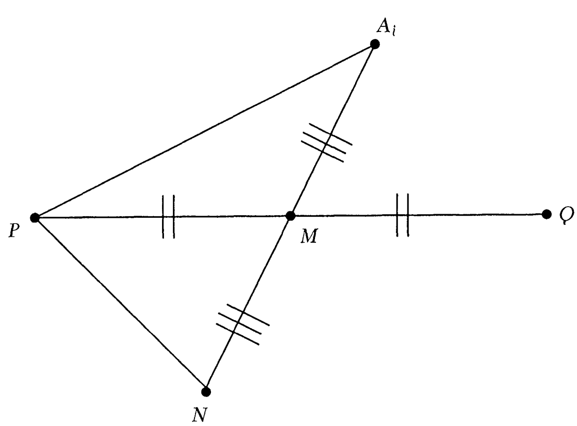

FIGURE 1
Constructing N.

Applying the triangle inequality to $\triangle A_iPN$ gives

$$2\|M - A_i\| < \|P - A_i\| + \|Q - A_i\|.$$

Summing over i gives

$$f(M) < \frac{f(P) + f(Q)}{2}.$$

Thus, the minimum of f cannot be attained at two different points P and Q. The unique point P that minimizes the function $f(X)$ is called the *Fermat point* of H.

THEOREM 3. *Let P be the Fermat point of a non-collinear n-hedron $H=\{A_1,\ldots,A_n\}$. Let the faces A_i^* and the vertex angles α_i of H be defined as above.*

(A) *If* $\cos\alpha_i > -1/(n-1)$ *for all* $i=1,\ldots,n$, *then P cannot be a vertex. In this case, P satisfies the equiangular property*

$$\cos\langle P; A_i^*\rangle = -1/(n-1) \quad \forall\, i=1,\ldots,n.$$

(B) *If* $\cos\alpha_i \le -1/(n-1)$ *for some values of i, then P must be a vertex. In this case, P is one of the vertices A_i for which* $\cos\alpha_i \le -1/(n-1)$.

Proof. We first suppose that P is not a vertex; we will show that

$$(*)\quad \cos\langle P; A_i^*\rangle = -1/(n-1); \qquad \text{and} \qquad (**)\quad \cos\alpha_i > -1/(n-1)$$

hold for all $i=1,\ldots,n$. This will prove the first half of (B) and the second half of (A).

Since the gradient

$$\nabla f(X) = \sum_{i=1}^{n} \frac{X-A_i}{\|X-A_i\|}$$

of f exists at every point in $\mathbb{R}^m$ except at the vertices $A_1,\ldots,A_n$ of H, it follows that $\nabla f(P)=0$. Thus

$$\sum_{i=1}^{n} P_i = 0, \quad \text{where} \quad P_i = \frac{A_i - P}{\|A-P_i\|}.$$

Dotting with $P_1,\ldots,P_n$, we obtain

$$1 + \sum_{j=1, j\neq i}^{n} P_i\cdot P_j = 0, \quad i=1,\ldots,n.$$

Since $P_i\cdot P_j = \cos\angle A_iPA_j$, it follows that

$$\sum_{j=1, j\neq i}^{n} \cos\angle A_iPA_j = -1, \quad i=1,\ldots,n. \tag{2}$$

Adding Equations (2) gives $n + 2\sum_{1\le i<j\le n}\cos\angle A_iPA_j = 0$. Therefore,

$$\begin{aligned}
-\frac{n}{2} &= \sum_{1\le i<j\le n} \cos\angle A_iPA_j \\
&= \left[\sum_{1\le i<j\le n-1} \cos\angle A_iPA_j\right] + \left[\sum_{1\le j\le n-1} \cos\angle A_nPA_j\right] \\
&= \frac{(n-1)(n-2)}{2}\cos\langle P;\{A_1,\ldots,A_{n-1}\}\rangle + (-1) \quad (\text{from (2) and (1)}) \\
&= \frac{(n-1)(n-2)}{2}\cos\langle P; A_n^*\rangle - 1.
\end{aligned}$$

Hence it follows that

$$\cos\langle P;\, A_n^* \rangle = \frac{2(1-n/2)}{(n-1)(n-2)} = -\frac{1}{n-1}.$$

Similarly, one sees that every $(n-1)$-hedral subangle $\langle P;\, A_i^* \rangle$ of $\langle P;\, H \rangle$ has cosine $-1/(n-1)$. Therefore P satisfies the equiangular property $(*)$.

To establish $(**)$, let Q be any point on the extension of the segment $A_n P$. (See FIGURE 2.)

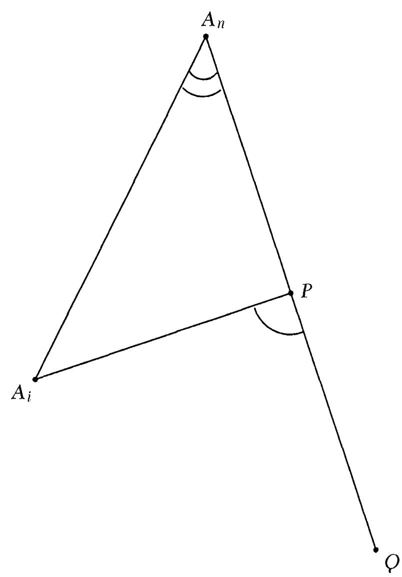

FIGURE 2
Bounding the cosine.

Then

$$\begin{aligned}
&\left[(n-1)+(n-1)(n-2)\cos\alpha_n\right]^{1/2} \\
&\qquad \geq \sum_{i=1}^{n-1} \cos\angle A_i A_n Q \qquad \text{(by Lemma 1)} \\
&\qquad > \sum_{i=1}^{n-1} \cos\angle A_i P Q \qquad \text{(exterior angle theorem)} \\
&\qquad = \sum_{i=1}^{n-1} -\cos\angle A_i P A_n \qquad (\text{since } \cos(\pi-\beta) = -\cos\beta) \\
&\qquad = 1 \qquad (\text{by } (2)).
\end{aligned}$$

Therefore $n-1+(n-1)(n-2)\cos\alpha_n > 1$, and hence $\cos\alpha_n > -1/(n-1)$; similar statements hold for every α_i. This proves $(**)$ and consequently settles both the first half of (B) and the second half of (A).

To complete the proof, we suppose that H has the property $\cos\alpha_n > -1/(n-1)$; we will show that the Fermat point P cannot coincide with the vertex A_n. Without loss of generality we may assume that A_n is the origin O.

Let X be the unit vector that maximizes $\Sigma_{i=1}^{n-1}\cos\angle XOA_i$ (see Lemma 1). Then

$$\begin{aligned}\sum_{i=1}^{n-1}\cos\angle XOA_i &= \left[(n-1)+(n-1)(n-2)\cos\alpha_n\right]^{1/2} && \text{(by Lemma 1)}\\ &> \left[(n-1)+(n-1)(n-2)(-1/(n-1))\right]^{1/2} && \\ & && \text{(by our assumption on } \cos\alpha_n)\\ &= 1. && (3)\end{aligned}$$

For $t \in \mathbb{R}$, define $f(t) = \Sigma_{i=1}^{n}\|tX - A_i\|$. Then

$$f(t) = \|tX - A_n\| + \sum_{i=1}^{n-1}\|tX - A_i\| = |t| + \sum_{i=1}^{n-1}\|tX - A_i\|,$$

and, for $t \geq 0$, we have

$$f'(t) = 1 + X\cdot\sum_{i=1}^{n-1}\frac{tX - A_i}{\|tX - A_i\|},$$

where $f'(0)$ denotes the right-hand derivative of f at 0. Hence, from (3),

$$f'(0) = 1 - X\cdot\sum_{i=1}^{n-1}\frac{A_i}{\|A_i\|} = 1 - \sum_{i=1}^{n-1}\cos\angle XOA_i < 0.$$

Thus f cannot attain its minimum at $t = 0$. Equivalently, A_n cannot be the Fermat point of H. Similar statements hold for the other vertices A_i. This completes the proof.

Part (B) of the preceding theorem suggests that a given n-hedron $H = \{A_1, \dots, A_n\}$ may have more than one vertex A_i at which $\cos\alpha_i \leq -1/(n-1)$. The next theorem shows that for triangles and tetrahedra, this possibility does not arise. (For hedra of higher dimensions, the situation remains open.)

THEOREM 4. *Let $H = \{A_1, \dots, A_n\}$ be a given non-collinear n-hedron. If $n = 3$ or 4, then H has at most one vertex A_i for which* $\cos\alpha_i \leq -1/(n-1)$.

Proof. For a triangle, this is easy, since $\cos\alpha_i \leq -1/2$ if and only if $\alpha_i = 120°$. For a tetrahedron $H = \{A_1, \dots, A_4\}$, one first observes that if a, b, and c are the angles of any triangle, then $1 \leq \cos a + \cos b + \cos c \leq 3/2$. Applying this to each face A_i^* of H, we conclude that the sum of the cosines of the twelve 2-hedral angles that can be formed from $\{A_1, \dots, A_4\}$ lies between 4 and 6. This shows that

$$4/3 \leq \cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3 + \cos\alpha_4 \leq 2.$$

Therefore, if $\cos\alpha_1 \leq -1/3$ and $\cos\alpha_2 \leq -1/3$, we would have

$$\cos\alpha_3 + \cos\alpha_4 \geq \frac{4}{3} - \cos\alpha_1 - \cos\alpha_2 \geq 2.$$

But $\cos\xi \leq 1$ for all ξ, so $\cos\alpha_3 = \cos\alpha_4 = 1$. This contradicts the non-collinearity of $A_1, \dots, A_4$, and settles the case of a tetrahedron.

Open questions As mentioned above, it is unknown whether Theorem 4 holds for $n > 4$. If it does not hold, then for an n-hedron $H = \{A_1, \ldots, A_n\}$ having more than one vertex A_i with $\cos\alpha_i \le -1/(n-1)$, it would be interesting to find a method that locates the particular vertex that coincides with the Fermat point.

Another interesting open question concerns geometric characterizations of the Fermat points. It is well known ([**6**, pp. 24–34], [**2**, p. 22]) that if $H = \{A, B, C\}$ is a triangle all of whose angles are less than 120°, then the Fermat point P is characterized by the elegant geometric property of being the intersection of the three circles that circumscribe three equilateral triangles ABX, BCY, CAZ drawn outward on its sides. This characterization also describes a method of constructing P. For higher dimensional hedra, it is still unknown whether the Fermat points have similar geometric characterizations and simple methods for construction.

Still another interesting question was raised by Dr. Folke Eriksson. Let $\{A_1, \ldots, A_n\}$ be an n-hedron with a non-vertex Fermat point P, and let S be the unit solid sphere centered at P. For $j = 1, \ldots, n$, let S_j be the part of S enclosed in the $(n-1)$-hedral angle $\langle P; A_j^* \rangle$. Thus S_j is the set of all points of S that can be written in the form $\Sigma r_i(PA_i)$, with $r_j = 0$ and $r_i \ge 0$, $\forall i \ne j$. Then the equiangular property of P can be stated, when $n = 3$ or 4, in the following three equivalent ways:

(i) $\cos\langle P; A_1^* \rangle = \cos\langle P; A_2^* \rangle = \cdots = \cos\langle P; A_n^* \rangle$
(ii) $S_1, S_2, \ldots, S_n$ have the same volumes
(iii) $S_1, S_2, \ldots, S_n$ are P-congruent (P-isometric).

In higher dimensions, we have seen that P must satisfy (i). Whether it has to satisfy (ii) and (iii) is still unknown. Dr. Eriksson conjectures that (iii) need *not* be satisfied by a Fermat point, and also raises an interesting question related to (ii): Given n linearly independent unit vectors $u_1, u_2, \ldots, u_n$ in $\mathbb{R}^n$, what is the n-dimensional volume (or Lebesgue measure) of the set

$$\left\{x = \sum c_j u_j \in \mathbb{R}^n : \|x\| \le 1,\ c_j \ge 0\right\}?$$

Acknowledgment. The authors thank Professor Folke Eriksson for drawing their attention to references [**7**] and [**4**] and for providing an English translation of the essential parts of [**4**].

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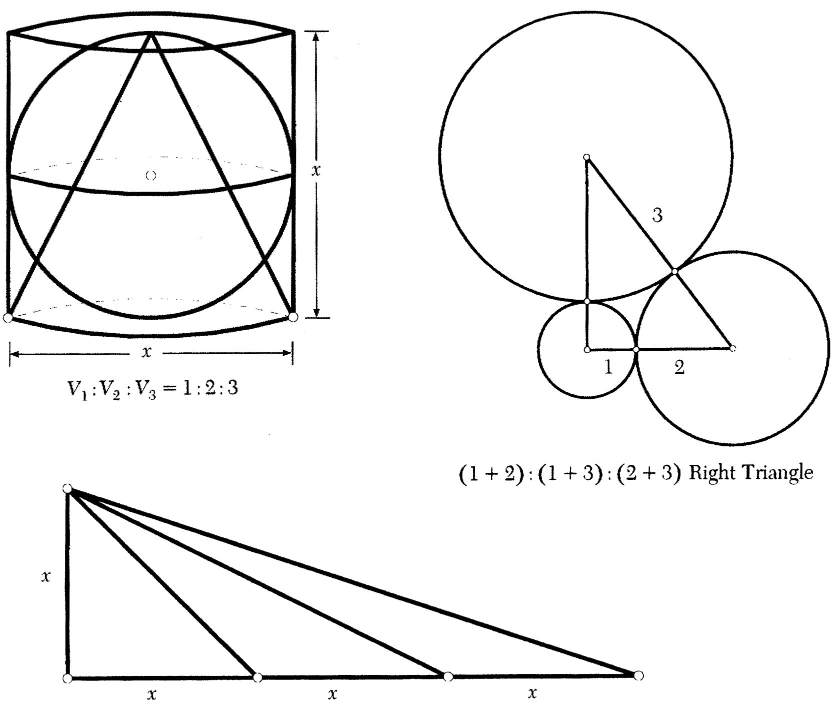
Math Bite: 1, 2, 3, Facetiae

$$1+2+3=3\cdot 2\cdot 1 \qquad 1!+2!+3!=3^{2^1}$$

$$(1+1i)(1+2i)(1+3i)=(1-1i)(1-2i)(1-3i)$$

$$\binom{1^2+2^2+3^2}{1+3}:\binom{1^2+2^2+3^2}{2+3}:\binom{1^2+2^2+3^2}{3+3}=1001:2002:3003=1:2:3^*$$



$V_1:V_2:V_3=1:2:3$

$(1+2):(1+3):(2+3)$ Right Triangle

$\cot^{-1}1=\cot^{-1}2+\cot^{-1}3$

FIGURE 1

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1^2			2^2					3^2									3^2					2^2			1^2

W I Z A R D

—Monte J. Zerger
Adams State College
Alamosa, CO 81102

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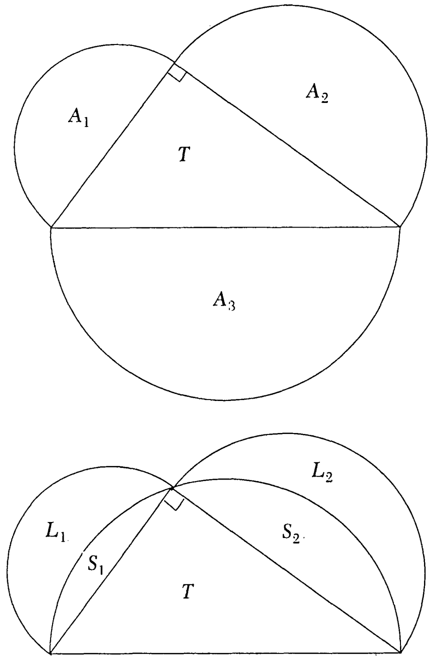
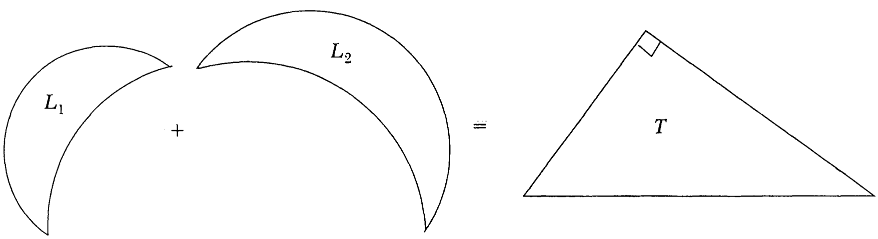
*Singmaster has shown that this is the only $\binom{n}{k}:\binom{n}{k+1}:\binom{n}{k+2}=1:2:3$. See D. Singmaster, Repeated binomial coefficients and Fibonacci numbers, *Fibonacci Quarterly* 13 (1975), 295–298.

Proof Without Words: Construction of Two Lunes with Combined Area Equal to That of a Given Right Triangle



$$A_1 + A_2 = A_3$$
$$(L_1 + S_1) + (L_2 + S_2) = T + S_1 + S_2$$
$$L_1 + L_2 = T$$



—EUGENE A. MARGERUM
—MICHAEL M. MCDONNELL
U. S. ARMY TOPOGRAPHIC ENGINEERING CENTER
7701 TELEGRAPH ROAD
ALEXANDRIA, VA 22315-3864

PROBLEMS

GEORGE T. GILBERT, *Editor*
Texas Christian University

ZE-LI DOU, KEN RICHARDSON, and SUSAN G. STAPLES, *Assistant Editors*
Texas Christian University

Proposals

To be considered for publication, solutions should be received by May 1, 1998.

1534. *Proposed by Donald Knuth, Stanford University, Stanford, California.*

Let m, n, and p be positive integers, and set

$$t_{m,p}(n) = \left\lceil \frac{\lfloor n/m \rfloor}{2p} \right\rceil, \qquad s_{m,p}(n) = t_{m,p}(0) + t_{m,p}(1) + \cdots + t_{m,p}(n-1).$$

Prove that $s_{m,p}(n)$ is a multiple of $t_{m,p}(n)$.

1535. *Proposed by Sergei Ovchinnikov, San Francisco State University, San Francisco, California.*

Let S be a nonempty set of real numbers. Prove that there exists a group G and a surjective function $f\colon G \to S$ satisfying

$$f(xy^{-1}) \geq \min\{f(x), f(y)\} \quad \text{for all } x, y \in G$$

if and only if $\sup S \in S$.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. *Each solution should begin on a separate sheet containing the solver's name and full address.*

Solutions and new proposals should be mailed to George T. Gilbert, Problems Editor, Department of Mathematics, Box 298900, Texas Christian University, Fort Worth TX 76129, or mailed electronically (ideally as a LATEX file) to `g.gilbert@tcu.edu`. *Readers who use e-mail should also provide an e-mail address.*

1536. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, New York.*

Let $c_n = \binom{2n}{n}/(n+1)$ be the Catalan numbers. Evaluate the determinants

$$A_n = \begin{vmatrix} 1 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ -1 & 1 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ 0 & -1 & 1 & \cdots & c_{n-4} & c_{n-3} \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & c_1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix} \quad \text{and}$$

$$B_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ -1 & 2 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ 0 & -1 & 2 & \cdots & c_{n-4} & c_{n-3} \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 2 & c_1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}.$$

1537. *Proposed by Jerrold W. Grossman, Oakland University, Rochester, Michigan.*

A two-person game is played as follows. A position consists of a pair (a, b) of positive integers. Players alternate moves, a move consisting of decreasing the larger number in the current position by any positive multiple of the smaller number, as long as the result remains positive. The first player unable to make a move loses. (This happens when $a = b$.) Determine those a and b such that the player who goes first from position (a, b) can guarantee a win with optimal play.

1538. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada, and George T. Gilbert, Texas Christian University, Fort Worth, Texas.*

Find all integer solutions to $2(x^5 + y^5 + 1) = 5xy(x^2 + y^2 + 1)$.

Quickies

Answers to the Quickies are on page 388.

Q871. *Proposed by Herbert Gülicher, Westfälische Wilhelms–Universität, Münster, Germany.*

In acute triangle ABC, erect external isosceles triangles $\triangle ABC'$, $\triangle BCA'$, and $\triangle CAB'$ such that $\angle ABC' = \angle BAC' = \angle ACB$, $\angle BCA' = \angle CBA' = \angle BAC$, and $\angle CAB' = \angle ACB' = \angle CBA$. Prove that AA', BB', and CC' are concurrent.

Q872. *Proposed by Charles Vanden Eynden, Illinois State University, Normal, Illinois.*

Show that $\cos(\cos x) \geq \sin x$ for all real numbers x.

Q873. *Proposed by Torsten Sillke, Lufthansa Systems GmbH, Frankfurt, Germany, and William P. Wardlaw, United States Naval Academy, Annapolis, Maryland.*

Suppose that A, B, C, and D are $n \times n$ matrices over a commutative ring such that $AC = CA$. Show that

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Solutions

Evaluation of a Permanent **December 1996**

1509. *Proposed by David Callan, University of Wisconsin, Madison, Wisconsin.*

Let A be a real $n \times n$ matrix satisfying (i) each row sums to 1; (ii) each entry immediately above the main diagonal is $1/2$; (iii) all other entries above the main diagonal are 0. Prove that the permanent of A is $1/2^{n-1}$.

(The permanent of a matrix is $\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$. Thus, it is similar in form to the determinant: $\sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)}$.)

I. *Solution by Philip D. Straffin, Beloit College, Beloit, Wisconsin.*

We prove the result by induction on n. It is clearly true for $n = 1$. Suppose it is true up through $(n-1) \times (n-1)$ matrices. Expanding in the first row and using linearity with respect to the first column, we obtain

$$\operatorname{perm}\begin{pmatrix} 1/2 & 1/2 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 1/2 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ & & & \ddots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

$$= \frac{1}{2}\operatorname{perm}\begin{pmatrix} a_{22} & 1/2 & 0 & \cdots & 0 \\ a_{32} & a_{33} & 1/2 & \cdots & 0 \\ & & & \ddots & \\ a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}$$

$$+ \frac{1}{2}\operatorname{perm}\begin{pmatrix} a_{21} & 1/2 & 0 & \cdots & 0 \\ a_{31} & a_{33} & 1/2 & \cdots & 0 \\ & & & \ddots & \\ a_{n1} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}$$

$$= \frac{1}{2}\operatorname{perm}\begin{pmatrix} a_{21}+a_{22} & 1/2 & 0 & \cdots & 0 \\ a_{31}+a_{32} & a_{33} & 1/2 & \cdots & 0 \\ & & & \ddots & \\ a_{n1}+a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}$$

$$= \frac{1}{2} \cdot \frac{1}{2^{n-2}} = \frac{1}{2^{n-1}},$$

since the last matrix is an $(n-1) \times (n-1)$ matrix satisfying the conditions.

II. *Solution by Tim Flood, Pittsburg State University, Pittsburg, Kansas.*

We will prove a more general statement. We replace condition (i) by

(i′) the bottom row sums to k and the remaining rows sum to 1.

Assuming the other hypotheses, we will show that the permanent of A is $k/2^{n-1}$; thus, the original result will follow.

We will prove this statement using induction on n. The permanent of (a_{11}) is $a_{11}/2^{1-1}$. We now assume the stronger statement holds for real $(n-1)\times(n-1)$ matrices and let A be a real $n\times n$ matrix satisfying these hypotheses. We will compute the permanent of A by expanding in the last column. We have

$$\operatorname{perm}(A)=\frac{1}{2}\operatorname{perm}(A_{n-1,n})+a_{n,n}\operatorname{perm}(A_{n,n}),$$

where $A_{i,j}$ is the submatrix of A created by deleting row i and column j. Clearly, $A_{n-1,n}$ and $A_{n,n}$ are $(n-1)\times(n-1)$ matrices that satisfy the hypotheses with bottom row sums $k-a_{n,n}$ and $1/2$, respectively. Thus, by the induction hypothesis, we have

$$\operatorname{perm}(A)=\frac{1}{2}\,\frac{k-a_{n,n}}{2^{n-2}}+a_{n,n}\frac{1/2}{2^{n-2}}=\frac{k}{2^{n-1}},$$

as required.

Also solved by Anchorage Math Solutions Group, Laura Batt, Richard Belshoff, J. C. Binz (Switzerland), Stan Byrd and Ronald L. Smith, Curtis Coker, Con Amore Problem Group (Denmark), L. L. Foster, Gerald A. Heuer, Murray S. Klamkin (Canada), Kee-Wai Lau (Hong Kong), F. C. Rembis, Edward Schmeichel, Zun Shan and Edward T. H. Wang (Canada), Achilleas Sinefakopoulos (student, Greece), William P. Wardlaw, Michael Woltermann, Yongzhi Yang, and the proposer. There was one incomplete solution.

Perfect Squares Within a Sequence — December 1996

1510. *Proposed by Detlef Laugwitz, Technische Hochschule Darmstadt, Darmstadt, Germany.*

Find the largest positive number C such that for every positive integer n, there is at most one perfect square in the set $\{1+k^2n: 2\le k\le c\sqrt{n}\}$.

Solution by Brian D. Beasley, Presbyterian College, Clinton, South Carolina.

We show that the largest such c is 8, even if one considers only those positive integers n that exceed some arbitrary lower bound.

Given $c>0$ and $n\in\mathbb{N}$, let $S_{c,n}=\{1+k^2n\colon 2\le k\le c\sqrt{n}\}$. We fix $c>8$ and find infinitely many n such that $S_{c,n}$ contains at least two squares. Because the sequence $((8j+4)/\sqrt{j(j+1)})_{j\ge 1}$ converges to 8, there exist infinitely many $j\in\mathbb{N}$ with $(8j+4)/\sqrt{j(j+1)}<c$. Taking $n=j(j+1)$, we have $8j+4<c\sqrt{n}$, so

$$1+(8j+4)^2n=(8j^2+8j+1)^2\in S_{c,n}.$$

Also, $1+2^2n=(2j+1)^2\in S_{c,n}$.

Next, we fix $n\in\mathbb{N}$ and show that $S_{8,n}$ contains at most one square. If n is a square, then $1+k^2n$ cannot be a square for $k\in\mathbb{N}$, so $S_{8,n}$ contains no squares. If n is not a square, then it is well-known that Pell's equation $x^2-nk^2=1$ has infinitely many positive integer solutions, all of which may be represented as

$$(x, k) = (x_1, k_1), (x_2, k_2), \ldots,$$

with $x_1 < x_2 < \cdots$ and $k_1 < k_2 < \cdots$. Moreover, $x_{m+1} = x_1 x_m + k_1 k_m n$ and $k_{m+1} = k_1 x_m + k_m x_1$. (See D. M. Burton, *Elementary Number Theory*, 3rd ed., 1997, McGraw-Hill, pp. 316–321). Thus we need only show that at most one k_m satisfies $2 \le k_m \le 8\sqrt{n}$. If $k_1 = 1$, then $n \ge 3$ since $k_1 = 2$ for $n = 2$. Thus,

$$k_3 = k_1 x_2 + k_2 x_1 = 1(2n+1) + (2\sqrt{1+n})(\sqrt{1+n}) = 4n + 3 > 8\sqrt{n}.$$

If $k_1 \ge 2$, then

$$k_2 = 2k_1 x_1 = 2k_1\sqrt{1 + nk_1^2} > 2k_1^2\sqrt{n} \ge 8\sqrt{n}.$$

Also solved by Anchorage Math Solutions Group, John Christopher, L. L. Foster, Dixon J. Jones, and the proposer.

Intersections of Convex Polygons **December 1996**

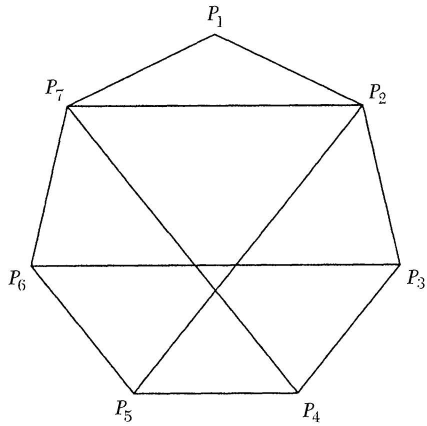
1511. *Proposed by "Ruby Rose Z″L," Pacific Lutheran University, Tacoma, Washington.*

Let $P_1, \ldots, P_7$ be 7 points in the plane. Consider the 35 convex polygons $\mathscr{P}_i$ formed by selecting 4 of the 7 points and taking their convex hull. Prove that:

(i) Among any 4 of the polygons $\mathscr{P}_i$ one can always find 3 that have a point in common.
(ii) There are 3 points in the plane such that every polygon contains at least one of the 3 points.
(iii) There are configurations of 7 points for which there do not exist 2 points such that every polygon contains at least one of the 2 points.

Solution by L. L. Foster, Northridge, California.

(i) Because there are only 7 vertices, the list of the 16 vertices from any 4 of the $\mathscr{P}_i$ must contain one of the vertices at least 3 times.
(ii) We claim that we may take P_6 and P_7 as two of the three desired points. There are 5 polygons $\mathscr{P}_i$ formed by $P_1, \ldots, P_5$ and two cases to consider. If one of $P_1, \ldots, P_5$ is in the convex hull of the other four points, we may take this P_i as our third point. If not, the convex hull of P_1, P_2, P_3, and P_4 must be a quadrilateral and the intersection of its diagonals must be contained in all 5 polygons.
(iii) Let the P_i be the vertices of a regular heptagon with center O, as in the diagram below. Suppose, on the contrary, that two such points A and B exist. If no P_i is on the line AB, then one side of AB must contain (at least) four of the P_i and hence their convex closure, a contradiction. Hence we may assume that P_1 is on the line AB. The convex quadrilaterals $P_1P_2P_3P_4$ and $P_1P_5P_6P_7$ intersect only at P_1. However, the points A, B, and P_1 cannot be collinear unless P_1 coincides with A or B, say $P_1 = A$. Since P_1 is not in the convex hull of $P_2P_3P_4P_5P_6P_7$, it follows that $B \in P_2P_3P_6P_7 \cap P_3P_4P_5P_6$; in other words, B is on the segment P_3P_6. Similar reasoning implies B is on the segments P_2P_5 and P_4P_7. However, by symmetry P_3P_6 and P_2P_5 intersect on the line OP_4 while P_2P_5 and P_4P_7 intersect on the line OP_1. This implies $B = O$, which is not in $P_3P_4P_5P_6$. The contradiction implies A and B do not exist.



Also solved by Philip D. Straffin and the proposer.

A Projection Property of a Function on Rings **December 1996**

1512. *Proposed by Arthur L. Holshouser, Charlotte, North Carolina, and Benjamin G. Klein, Davidson College, Davidson, North Carolina.*

Let R be a commutative ring such that $x^3 = x$ for every $x \in R$. For $x, y \in R$, let $F(x, y) = xy - x^2y - xy^2 - x^2y^2$. If $F(a, b) = a$ and $F(b, c) = b$, prove that $F(a, c) = a$.

I. *Solution by Irl C. Bivens, Davidson College, Davidson, North Carolina.*

We begin by deriving two identities that follow directly from the condition $x^3 = x$. First, because y^2 acts as an identity on any positive power of y, we have $y^2F(x, y) = F(x, y)$ for all x and y. Thus,

$$b^2a = b^2F(a,b) = F(a,b) = a,$$

so that $b^2a^2 = a^2$. Similarly, the equation $F(b,c) = b$ implies $c^2b^2 = b^2$.

Second, since

$$x + y = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x + 3x^2y + 3xy^2 + y,$$

it follows that $3(x^2y + xy^2) = 0$. Likewise, the equation $y - x = (y-x)^3$ implies $3(x^2y - xy^2) = 0$. Since $F(x, y)$ may be written in the form

$$F(x,y) = x(x^2y - xy^2) - (x^2y + xy^2),$$

these equations imply that $3F(x, y) = 0$ for all x and y. Therefore, $3a = 3F(a, b) = 0$ and $3b = 3F(b, c) = 0$. The polynomial $F(x, y)$ may be written in the form

$$F(x,y) = (x - x^2)(y - y^2) - 2x^2y^2,$$

so that

$$\begin{aligned} a = F(a,b) &= (a-a^2)(b-b^2) - 2a^2b^2 = (a-a^2)(b-b^2) + a^2b^2 \\ &= (a-a^2)(b-b^2) + a^2, \end{aligned}$$

or equivalently that $a - a^2 = (a - a^2)(b - b^2)$. Similarly, $F(b, c) = b$ implies $b - b^2 = (b - b^2)(c - c^2)$.

Since $c^2a^2 = c^2(b^2a^2) = (c^2b^2)a^2 = b^2a^2 = a^2$, the equation $F(a,c) = a$ may be written in the form $a - a^2 = (a - a^2)(c - c^2)$. But this equation is satisfied, since

$$\begin{aligned} a - a^2 &= (a - a^2)(b - b^2) = (a - a^2)\left[(b - b^2)(c - c^2)\right] \\ &= \left[(a - a^2)(b - b^2)\right](c - c^2) = (a - a^2)(c - c^2). \end{aligned}$$

II. *Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Define an operation $*$ by $x * y := F(x, y)$. Observe that $2x = (2x)^3 = 8x^3 = 8x$, hence $6x = 0$. From this, direct computation shows that $*$ is associative. Therefore,

$$F(a,c) = a * c = (a * b) * c = a * (b * c) = a * b = a.$$

Comment. Erwin Just pointed out that the condition $x^3 = x$ implies R is a commutative ring. This is a special case of a theorem of Jacobson. This case appears as an exercise in several undergraduate algebra books.

Also solved by Anchorage Math Solutions Group, Ron Martin Carroll, Con Amore Problem Group (Denmark), Timothy Davis (student), Joseph G. Gaskin, Erwin Just (emeritus), Kee-Wai Lau (Hong Kong), Paul Peck, Achilleas Sinefakopoulos (student, Greece), Jun Wang (student), and the proposers. There was one incorrect solution.

A Discrete Quartering Problem — December 1996

1513. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

Can every set of $4n$ points in the plane, no three of which are collinear, be evenly quartered by two mutually perpendicular lines?

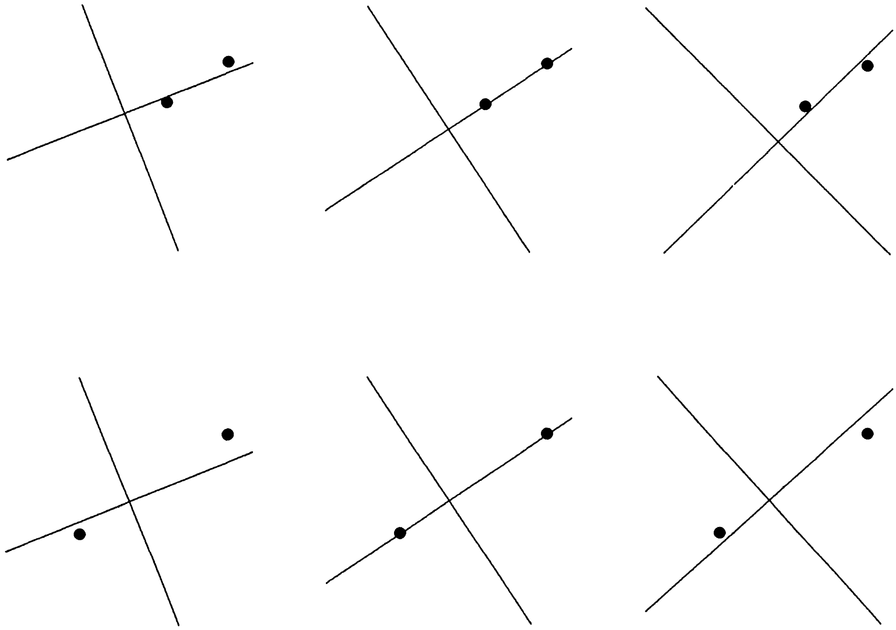
(The original, continuous version of this question appeared in Hugo Steinhaus' *One Hundred Problems in Elementary Mathematics*, Dover, 1979, 26.)

Solution by the proposer.

We shall prove that the result is true.

Let S denote the set of $4n$ points in the plane, no three of which are collinear. Given a directed line, the family of all directed lines in the plane with the same direction vector satisfies one of the following: (i) there exist infinitely many lines $\mathscr{L}$ in the family such that exactly $2n$ points in S are on each side of the line; (ii) there is a unique line $\mathscr{L}$ in the family that passes through two of the points of S and $2n - 1$ points in S are on each side of the line. In each case, we call such $\mathscr{L}$ *division lines*. There are only finitely many families that fall into case (ii). Furthermore, if one considers those families that intersect a given $\mathscr{L}$ satisfying (ii) in a sufficiently small angle, the division lines from one of these families satisfy (i) and they intersect the line segment between the two points of S on the given $\mathscr{L}$.

Fix a family satisfying (i) such that its two orthogonal families also satisfy (i). We parametrize families by the counterclockwise angle θ, $0 \leq \theta < 2\pi$, from the fixed family to the family. For all but finitely many angles, both the family and its orthogonal families satisfy (i). Denote the finite set of exceptional angles by A. For other angles θ, division lines corresponding to θ and to $\theta + \pi/2$ or $\theta - 3\pi/2$ may be thought of as x and y axes, with k points of S points in each of the 1st and 3rd quadrants and $2n - k$ points in each of the 2nd and 4th quadrants. Define a function f: $[0, 2\pi) - A \to \{0, 1, \dots, 2n\}$ by setting $f(\theta)$ equal to the number of 1st quadrant points. We must show there exists a θ with $f(\theta) = n$. Observe that f is locally constant, since f will not change under a small rotation of the axes. Equivalently, f is constant on each interval contained in $[0, 2\pi) - A$. Since $f(\pi/2) = 2n - f(0)$, it



suffices to prove that f cannot change by more than 1 at the angles in A. Consider an $\mathscr{L}$ from a family satisfying (ii), along with an $\mathscr{L}_\perp$ from one of the two orthogonal families, considering the pair to be the coordinate axes. If the two points of S on $\mathscr{L}$ (or $\mathscr{L}_\perp$) are on the same side of the "origin" (including the case when one point is the origin), they contribute nothing to the change in f as θ increases. If they are on opposite sides, f changes by ± 1 as θ increases. Thus, f could conceivably change by 2 only if there is a point on each positive and negative axis. However, this situation does not change f at all because one of the four points is in each quadrant for all sufficiently close angles θ.

There were two incorrect solutions.

Answers

Solutions to the Quickies on page 382.

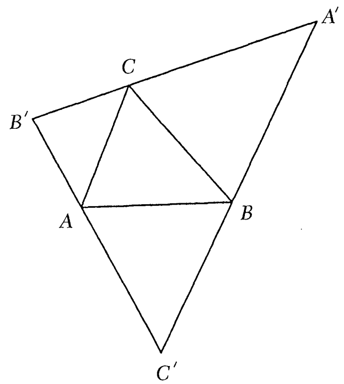
A871. By construction, A, B', C' are collinear, as are B, C', A' and C, A', B'. Also,

$$\frac{AC'}{C'B} = \frac{BA'}{A'C} = \frac{CB'}{B'A} = 1,$$

hence

$$\frac{A'C}{CB'} \cdot \frac{B'A}{AC'} \cdot \frac{C'B}{BA'} = 1.$$

Therefore, by Ceva's theorem, AA', BB', and CC' are concurrent (in the Gergonne point of $\triangle\ A'B'C'$).



A872. *I.* For $0 \leq x \leq \pi$, $\cos x$ is decreasing and $\sin x \leq x$. Thus, $\cos(\sin x) \geq \cos x$ in this range. Because $\cos(\sin x)$ and $\cos x$ are even functions, this latter inequality holds for $-\pi \leq x \leq \pi$, and hence for all x by periodicity. Therefore,

$$\cos(\cos x) = \cos(\sin(\pi/2 - x)) \geq \cos(\pi/2 - x) = \sin x$$

for all x.

II. *Provided by the Editors.* Because $|\cos x| \leq 1$, it follows from substituting $\cos x$ into the Maclaurin series for $\cos x$ that

$$\cos(\cos x) \geq 1 - \frac{\cos^2 x}{2!} = \frac{1 + \sin^2 x}{2} \geq \sin x.$$

A873. Without loss of generality, we may assume the ring R has a multiplicative identity 1. Let $A(x) = xI + A$ over the polynomial ring $R[x]$. Observe that $A(x)C = CA(x)$ and that $\det A(x)$ is a monic polynomial of degree n, hence is not a zero divisor in $R[x]$. Then

$$\begin{aligned}\det A(x) \cdot \det\begin{pmatrix} A(x) & B \\ C & D \end{pmatrix} &= \det\begin{pmatrix} I & 0 \\ -C & A(x) \end{pmatrix} \cdot \det\begin{pmatrix} A(x) & B \\ C & D \end{pmatrix} \\ &= \det\begin{pmatrix} A(x) & B \\ 0 & A(x)D - CB \end{pmatrix} \\ &= \det A(x) \cdot \det(A(x)D - CB).\end{aligned}$$

Because $\det A(x)$ is not a zero divisor, cancellation yields

$$\det\begin{pmatrix} A(x) & B \\ C & D \end{pmatrix} = \det(A(x)D - CB).$$

Substituting $x = 0$ gives the desired identity.

Acknowledgments. The editors would like to thank Murray S. Klamkin, Loren C. Larson, Efton Park, Daniel H. Ullman, and Peter Yff for their help in reviewing problem proposals over the last two years.

REVIEWS

PAUL J. CAMPBELL, editor
Beloit College
1997–98: University of Augsburg, Germany

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Chaitin, Gregory J., *The Limits of Mathematics: A Course on Information Theory and the Limits of Formal Reasoning*, Springer-Verlag, $29.95. ISBN 0–981–3083–59–X.

"Capture a piece of mathematics history-in-the-making," proclaims the publisher's hyperbole for this book. (If you want to capture it in electronic form instead, check out `http://www.cs.auckland.ac.nz/CDMTCS/chaitin/lm.html`.) Chaitin, at the IBM Watson Research Center, is the successor to Gödel in exploring what mathematics can't do. His main contribution is to consider incompleteness results in the light of information theory (*algorithmic information theory*). For example, call a computer program *elegant* if no smaller program in that language has the same output; certainly there are infinitely many elegant programs. Chaitin embeds into the programming language LISP a formal axiomatic system for reasoning about programs (much as Gödel embedded into arithmetic a statement that says of itself that it is unprovable), then shows that such a system can't prove the elegance of programs much larger than itself—hence can prove the elegance of at most finitely many programs. "Most people believe that anything that is true is true for a reason [i.e., that there is a proof]. These theorems show that some things are true for no reason at all, i.e., accidentally, or at random [because there is no proof]." This book contains edited transcripts of four lectures by Chaitin. The first is understandable by the general reader; but the others require basic knowledge of the computer language LISP, and readers can try Chaitin's LISP programs and LISP interpreter (written in Mathematica).

GIMPS Discovers 36th Known Mersenne Prime, $2^{2976221} - 1$ is now the Largest Known Prime. Under "What's New in Mathematics" at the AMS web site, `http://www.ams.org` .

GIMPS (Great Internet Mersenne Prime Search) uses spare time 2,000 volunteers' computers to search for new Mersenne primes (primes of the form $2^p - 1$, with p prime). Each volunteer's computer tries a different set of candidates. The volunteer lucky to be assigned the "winning" number this time was Gordon Spence, an information technology manager at a British electronics company, whose 100 MHz Pentium computer took 15 days to establish primality. You too can join the hunt; the necessary software is available at `http://www.mersenne.org/prime.htm` . But what are Mersenne primes good for? Apart from testing hardware and stimulating the finding of faster ways to multiply, they are now used in an encryption system based on elliptic curves. The GIMPS program itself is used by Intel to test Pentium chips before they ship. (In July another joint effort, by 14,000 computers, decoded a message in the U.S. commercial 56-bit DES code, using a brute-force attack on the 72 quadrillion possible keys and succeeding half-way through.)

Tufte, Edward R., *Visual Explanations: Images and Quantities, Evidence and Narrative*, Graphics Press, 1997; 158 pp, $45. ISBN 0–9613921–2–6. Wainer, Howard, *Visual Revelations: Graphic Tales of Fate and Deception from Napoleon Bonaparte to Ross Perot*, Springer-Verlag, 1997; xi + 180 pp, $33.95. ISBN 0–387–94902–X.

Author Tufte, author of two previous beautiful books that make a point about presentation of information in graphical form (*The Visual Display of Quantitative Information*, *Envisioning Information*), here focuses on "design strategies—the proper arrangement in space and time of images, words, and numbers—for presenting information about motion, process, mechanism, cause and effect." He suggests that his books treat pictures of respectively numbers, nouns, and (in this new one) verbs. His analysis of the poor graphics devised by engineers opposing the Space Shuttle *Challenger launch* is riveting and unforgettable. Wainer's book, which consists of edited versions of his columns in *Chance* magazine, contains fewer examples from art and architecture than Tufte's, instead concentrating on graphs and charts from newspapers and magazines. Wainer treats graphical failures and successes (his own fantasized presentation to a tobacco company is a classic), the various types of plots, and how to improve graphical presentations. Both books are clear, convincing, and beautiful, honoring the adage, "As for a picture, if it isn't worth a thousand words, the hell with it" (quoted by Tufte from Ad Reinhardt).

Beyond Numbers. Permanent and traveling exhibition at the Maryland Science Center.

Museum exhibits about mathematics are rare. This new one in Baltimore, jointly designed by Museum staff and mathematicians from George Washington University, gets away from the tired equation "math = numbers." Instead, it exhibits mathematics as a science of patterns, a discipline of solving problems, and an art of creating imaginative abstractions. Mathematicians attending the Joint Mathematics Meetings this January will find the Maryland Science Center just a short walk from the location of the meetings; others can find the schedule of the traveling exhibition at `http://www.ams.org` (there are open dates).

Morgan, Frank, Math Chat. Weekly column in the *Christian Science Monitor*, also available at the MAA Web site `http://www.maa.org`.

For over a year, Frank Morgan of the Mathematics Dept. at Williams College has been writing a weekly column about mathematics. The format is first answers from readers to an old challenge, then remarks on various topics, and finally a new challenge or two. Recent topics have included "the Bible code" and coincidences (with a contest for the least likely true coincidence), the thickness of folded paper, *Flatland*, towers of powers of 5s, and the number of distinct rearrangements of the letters in `HOMEGAME`. Even if there are some nonmathematical topics (the most efficient way to accelerate onto the freeway) or if the mathematics discussed is not the latest about the Riemann hypothesis, this column addresses an interest of the public (to relate to mathematics) and a need of our profession (to relate to the public).

Rae-Dupree, Janet, New programs seek patterns: Businesses are realizing that big dollars are hidden in their bulging databanks, *San Jose Mercury News* (6 October 1997).

Retail stores routinely collect data on customers' buying habits; this "transaction processing" collects immense amounts of data. Businesses' new hot hope is that automated "data mining" of these "data warehouses" will reveal hidden patterns to inform decisions and enhance profits. Data-mining software examines for associations, clusters, sequential patterns, similar time series, and classifications. Typical uses are to classify credit applicants as good or bad risks and to offer "individualized" incentives or coupons to customers; the typical underlying mathematical techniques are regression, correlation, and neural networks. But are the patterns that the software "discovers" really there?

NEWS AND LETTERS

Letter to the Editor

Dear Editor:

I read with interest the article *Thales Meets Poincaré*, by David Dobbs, in the June 1997 issue of *Mathematics Magazine*. The analytic approach taken by the author, which he says relies on "real analytic functions, as studied in advanced calculus," obscures, I think, the underlying geometric picture with long, complex-looking, numerical calculations. It finds a one-parameter family of counterexamples to Thales. If one takes the synthetic approach to hyperbolic geometry, one can show without the use of any metric that in the triangle ABC, where L, M, and N are the midpoints of the corresponding sides, the segment NM *cannot* be congruent to the segment BL. Every triangle is thus a counterexample to Thales theorem. The proof uses only the axioms of congruence and the fact that in hyperbolic geometry the angle sum of any triangle is less than two right angles. The problem appears as an exercise for the student (with hints!) in *Euclidean and non-Euclidean Geometries*, Marvin Jay Greenberg, 2nd ed., 1980, W.H. Freeman, page 164, #9.

The second finding in the paper, namely that in the limit (as the sides of the triangle become very small) Thales theorem becomes valid, is again demonstrated by a very specific example of a triangle with the lengths of the sides chosen very carefully. Even with this choice, the computations look daunting. It is, however, true for any right triangle with sides of length a, b, and c in the Poincaré model that $\cosh c = \cosh a \cosh b$. See, for instance, page 334 of the book cited above. If the hypotenuse c approaches zero, so do the other two sides. Using the well-known Maclaurin expansion for cosh and very small a, b, and c, this translates to $1 + c^2 \approx (1 + a^2)(1 + b^2)$ or $c^2 \approx a^2 + b^2$. In other words, for very small right triangles, the Pythagorean theorem is approximately true. This is enough to establish that in the limit, hyperbolic geometry approaches Euclidean geometry and that therefore Thales theorem will hold almost exactly in very small triangles.

I am always happy to see articles that may interest students in geometry, but my bias is towards using pure geometry wherever it provides a simple approach without numerical computation. Students have enough of the computational side of math elsewhere.

Dr. Clare Friedman
University of San Diego
San Diego, CA 92110-2492

The author replies:

Dr. Friedman's first criticism seems to be that I did not write the paper she wanted. The introduction to *Thales Meets Poincaré* (TMP, for short) states that the paper is intended for "model-oriented courses on absolute geometry." As developed in TMP, the Example depends only on the basics of the Poincaré model and is thus immediately accessible. By contrast, the synthetic proof cited by Dr. Friedman is sketched on page 164 of the Greenberg text, and depends on the Lagrange-Saccheri result which goes well beyond the axiomatic bases of the synthetic method.

Dr. Friedman's second criticism seems vague to me. One who has read the proof of the Theorem in TMP using the Poincaré model and its metric might give some meaning to her sentiment that "in the limit, hyperbolic geometry approaches Euclidean geometry." However, her meaning does not come through clearly in her brief comment. In any case, when it comes to an analytic approach, I would prefer that of TMP via power series (which are part of every mathematician's arsenal) to Dr. Friedman's argument using hyperbolic cosines, since many universities fail to teach the hyperbolic trigonometric functions in calculus courses. As in the preceding paragraph, I believe that the methods used in TMP are more accessible than those proposed by Dr. Friedman.

Exhortive terms such as "pure geometry" and "simple approach" are colorful but empty. We need to be more open-minded, accommodating broader methodologies and recognizing where others are coming from. Model-oriented courses are a fact of life, as is the state of student readiness for them at the junior level. TMP demonstrates that power series are the right tool to develop certain themes for the intended audience. In particular, TMP illustrates that models make possible quantitative studies which can give precise meaning to vague, intuitive statements such as the quotation in the preceding paragraph. While the computations with power series in TMP may "look daunting" to Dr. Friedman, I would hope any such impression is dispelled for any reader who looks beyond first impressions.

David E. Dobbs
The University of Tennessee
Knoxville, TN 37996-1300

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Wong, Roman, *Washington & Jefferson College, Washington, PA*

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Alavi, Yousef, *Western Michigan University, Kalamazoo, MI*

Althoen, Steven, *University of Michigan, Flint, MI*

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Goldstein, Jerome, *University of Memphis, Memphis, TN*

Growney, Joanne, *Bloomsburg University of Pennsylvania, Bloomsburg, PA*

Grunbaum, Branko, *University of Washington, Seattle, WA*

Guichard, David, *Whitman College, Walla Walla, WA*

Gulick, Denny, *University of Maryland, College Park, MD*

Guy, Richard, *University of Calgary, Calgary, Alberta, Canada*

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Kreminski, Richard, *Texas A&M University, Commerce, TX*

Krusemeyer, Mark, *Carleton College, Northfield, MN*

Kung, Sidney, *Jacksonville University, Jacksonville, FL*

Kuzmanovich, James, *Wake Forest University, Winston-Salem, NC*

Larson, Loren C., *St. Olaf College, Northfield, MN*

Lautzenheiser, Roger, *Rose-Hulman Institute of Technology, Terre Haute, IN*

Liu, Andrew, *University of Alberta, Edmonton, Alberta, Canada*

LoBello, Anthony, *Allegheny College, Meadville, PA*

Long, Calvin, *Northern Arizona University, Flagstaff, AZ*

McCleary, John, *Vassar College, Poughkeepsie, NY*

McDermot, Richard, *Allegheny College, Meadville, PA*

Merrill, Kathy, *Colorado College, Colorado Springs, CO*

Needham, Tristan, *University of San Francisco, San Francisco, CA*

Nelsen, Roger, *Lewis and Clark College, Portland, OR*

Nunemacher, Jeffrey, *Ohio Wesleyan University, Delaware, OH*

O'Shea, Donal, *Mount Holyoke College, South Hadley, MA*

Otero, Daniel, *Xavier University, Cincinnati, OH*

Parry, Cyril, *Exmouth, Devon, England*

Pedersen, Jean, *Santa Clara University, Santa Clara, CA*

Perline, Ronald, *Drexel University, Philadelphia, PA*

Reid, James D., *Wesleyan University, Middletown, CT*

Reznick, Bruce, *University of Illinois, Urbana, IL*

Richey, Matthew, *St. Olaf College, Northfield, MN*

Richmond, Thomas, *Western Kentucky University, Bowling Green, KY*

Riley, John, *Bloomsburg University of Pennsylvania, Bloomsburg, PA*

Ross, Kenneth, *University of Oregon, Eugene, OR*

Rothman, Neal, *Indiana University–Purdue University, Indianapolis, IN*

Roy, Ranjan, *Beloit College, Beloit, WI*

Sandefur, James, *Georgetown University, Washington, DC*

Saxe, Karen, *Macalester College, St. Paul, MN*

Schattschneider, Doris, *Moravian College, Bethlehem, PA*

Schoenfeld, Alan, *University of California, Berkeley, CA*

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THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, NW
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